MA3220 Ordinary Differential Equations

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Reference books:

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1. First-Order Differential Equations

1.1. *Introduction*

Definition 1.1 (ordinary differential equation). An ordinary differential equation (ODE) is a relation containing one real variable x, the real dependent variable y, and some of its derivatives $y', y'', \ldots, y^{(n)}, \ldots$ with respect to x.

The order of an ODE is defined to be the order of the highest derivative that occurs in the equation. As such, an n^{th} order ODE has the general form

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0.$$

Definition 1.2 (linear ODE). An n^{th} order ODE is said to be linear if it can be written in the form

$$a_0(x) y^{(n)} + a_1(x) y^{(n-1)} + \ldots + a_n(x) y = r(x).$$

The functions $a_j(x)$, where $0 \leq j \leq n$, are called the coefficients of the equation. Also, we will always assume that $a_0(x) \neq 0$ in any interval for which the equation is defined.

If $r(x) = 0$, the equation is said to be homogeneous. The equation is said to be non-homogeneous otherwise, and $r(x)$ is called the non-homogeneous term.

A functional relation between the dependent variable y and the independent variable x that satisfies the given ODE in some interval is called a solution of the given ODE on the interval. A general solution of an n^{th} order ODE depends on n arbitrary constants. Recall the equation mentioned in Definition 1.1, so a first-order ODE may be written as

$$F\left(x, y, y'\right) = 0.$$

In this chapter, we will only consider first-order ODE.

Definition 1.3 (explicit solution). The function $y = \phi(x)$ is called an explicit solution of $F(x, y, y') = 0$ in the interval J provided that $F(x, \phi(x), \phi'(x)) = 0$ for all $x \in J$.

Definition 1.4 (implicit solution). A relation of the form $\psi(x, y) = 0$ is an implicit solution of $F(x, y, y') = 0$ provided it determines one or more functions $y = \phi(x)$ which satisfy $F(x, \phi(x), \phi'(x)) = 0$.

Definition 1.5 (parametric solution). The pair of equations

$$x = x(t) \quad \text{and} \quad y = y(t)$$

is a parametric solution of $F(x, y, y') = 0$ if

$$F\left(x(t), y(t), \frac{\dot{y}(t)}{\dot{x}(t)}\right) = 0.$$

Example 1.1. Consider the ODE

$$x + y\frac{dy}{dx} = 0 \quad \text{for} \quad x \in (-1, 1).$$

We say that $x^2 + y^2 = 1$ is an implicit solution, whereas $x = \cos t$ and $y = \sin t$, where $t \in (0, \pi)$, is a parametric solution.

The solutions of a first-order ODE

$$\frac{dy}{dx} = f(x, y)$$

represent a one-parameter family of curves in the xy-plane. These are called integral cuvres. In other words, if $y = y(x)$ is a solution to $y' = f(x, y)$, then the vector field $\mathbf{F}(x, y) = \langle 1, f(x, y) \rangle$ is tangent to the curve $\mathbf{r}(x) = \langle x, y(x) \rangle$ at every point (x, y) since $\mathbf{r}'(x) = \mathbf{F}(x, y)$.

Given a family of functions parametrised by some constants, a differential equation can be formed by eliminating the constants of this family and its derivatives. Consider Example 1.2 for instance.

Example 1.2. The family of functions $y = Ae^x + B\sin x$ satisfies the ODE

$$\frac{d^4 y}{dx^4} - y = 0$$

when the constants A and B are eliminated using the derivatives.

Example 1.3. Find the differential equation satisfied by the family of functions

$$y = x^c \quad \text{for } x > 0, \text{ where } c \text{ is a parameter.}$$

Solution. We have

$$\ln y = c \ln x \quad \text{so} \quad \frac{1}{y}\frac{dy}{dx} = \frac{c}{x}.$$

As such,

$$\frac{dy}{dx} = \frac{y}{x} \cdot \frac{\ln y}{\ln x} = \frac{y \ln y}{x \ln x},$$

which is the desired differential equation. □

Definition 1.6 (separable equation). A separable differential equation can be written as

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}.$$

It is easy to obtain the solution of a separable differential equation (Definition 1.6). By rearranging the equation, we have

$$g(y)\ dy = f(x)\ dx.$$

Integrating both sides with respect to their respective variables, we obtain the general solution.

Example 1.4. Solve

$$\frac{dy}{dx} = -2xy \quad \text{with the initial condition } y(0) = 1.$$

Solution. This is a simple exercise from MA2002 so we omit the solution. One can deduce that $y = e^{-x^2}$. □

More generally, given the differential equation

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right),$$

we can reduce it to a separable equation by using the substitution $u = y/x$. As such, we obtain the differential equation

$$\frac{1}{f(u) - u}\,du = \frac{1}{x}\,dx.$$

Example 1.5. Solve

$$2xy\frac{dy}{dx} + x^2 - y^2 = 0.$$

Solution. Dividing both sides by x^2, we obtain

$$2\left(\frac{y}{x}\right)\frac{dy}{dx} + 1 - \left(\frac{y}{x}\right)^2 = 0.$$

This prompts us to consider the substitution $u = y/x$, so

$$\frac{du}{dx} = \frac{x\frac{dy}{dx} - y}{x^2} = \frac{1}{x}\left(\frac{dy}{dx} - u\right).$$

The differential equation becomes

$$2u\left(x\frac{du}{dx} + u\right) + 1 - u^2 = 0 \quad \text{so} \quad u^2 + 2xu\frac{du}{dx} + 1 = 0.$$

The remaining process is trivial. □

Definition 1.7 (homogeneous function)**.** A function is said to be homogeneous of degree n if

$$f(tx, ty) = t^n f(x, y) \quad \text{for all } x, y, t.$$

Example 1.6. $\sqrt{x^2 + y^2}$ and $x + y$ are homogeneous of degree 1, $x^2 + y^2$ is homogeneous of degree 2, and $\sin(x/y)$ is homogeneous of degree 0.

Definition 1.8 (homogeneous DE)**.** The ODE

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

is said to be homogeneous of degree n if both $M(x, y)$ and $N(x, y)$ are homogeneous of degree n.

If we write the differential equation

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

as

$$\frac{dy}{dx} = f(x, y) \quad \text{where} \quad f(x, y) = -\frac{M(x, y)}{N(x, y)},$$

then $f(x, y)$ is homogeneous of degree 0. To solve the DE $y' = f(x, y)$, we consider the substitution $y = xz$ so the differential equation becomes

$$z + x\frac{dz}{dx} = f(x, xz) = x^0 f(1, z) = f(1, z).$$

As such, the variables can be separated to yield

$$\frac{1}{f(1,z)-z}\,dz=\frac{1}{x}\,dx.$$

Integrating both sides yields the solution.

Example 1.7. For example, we wish to solve the differential equation

$$\frac{dy}{dx}=\frac{x+y}{x-y}.$$

Solution. Using the substitution $y=zx$, we have

$$\begin{aligned} z+x\frac{dz}{dx}&=\frac{x+zx}{x-zx}\\ x\frac{dz}{dx}&=\frac{1+z}{1-z}-z\\ \frac{dz}{dx}&=\frac{1+z^2}{x(1-z)} \end{aligned}$$

Using separation of variables, and integrating both sides,

$$\begin{aligned} \int\frac{1-z}{1+z^2}\,dz&=\int\frac{1}{x}\,dx\\ \tan^{-1}(z)-\frac{1}{2}\ln(1+z^2)&=\ln|x|\\ \tan^{-1}\left(\frac{y}{x}\right)&=\ln\sqrt{x^2+y^2}+c \end{aligned}$$

□

An equation of the form

$$\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$$

can be reduced to a homogeneous equation by a suitable substitution

$$x=z+h \text{ and } y=w+k \quad \text{where} \quad a_1b_2\neq a_2b_1,$$

where h and k are solutions to the system of linear equations

$$a_1h+b_1k+c_1=0 \quad \text{and} \quad a_2h+b_2k+c_2=0.$$

To see why this works, consider the original differential equation. The numerator can be written as

$$a_1(z+h)+b_1(w+k)+c_1=a_1z+b_1w+c_1,$$

whereas the denominator can be written as

$$a_2(z+h)+b_2(w+k)+c_2=a_2z+b_2w+c_2.$$

Note that

$$\frac{dx}{dz}=1 \text{ and } \frac{dy}{dw}=1 \implies \frac{dy}{dx}=\frac{dw}{dz}.$$

As such, the differential equation can be written as

$$\frac{dw}{dz}=\frac{a_1z+b_1w+c_1}{a_2z+b_2w+c_2}.$$

This new equation will become easier to solve.

1.2. *Exact Equations and Integrating Factors*

Definition 1.9 (exact ODE). We can write a first order ODE in the following form:

$$M(x,y)dx + N(x,y)dy = 0$$

This differential equation is exact if there exists a function $u(x,y)$ such that

$$M(x,y)dx + N(x,y)dy = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

Proposition 1.1. If the differential equation is exact, then the general solution is $u(x,y) = c$, where c is an arbitrary constant.

Theorem 1.1 (condition for ODE to be exact). If we have a first order ODE of the form $M(x,y)dx + N(x,y)dy = 0$ and we assume M and N, together with their first partial derivatives, to be continuous in

$$\text{the rectangle } S \quad \text{defined by the region } |x - x_0| < a \text{ and } |y - y_0| < b,$$

a necessary and sufficient condition for the differential equation to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{for all } (x,y) \in S.$$

Note that this has some semblance to Green's theorem. When this condition is satisfied, a general solution, given by $u(x,y) = c$, can be written as

$$u(x,y) = \int_{x_0}^{x} M(s,y)\, ds + \int_{y_0}^{y} N(x_0,t)\, dt \quad \text{for some constant } c.$$

Example 1.8. We wish to solve the differential equation

$$(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0.$$

Solution. This differential equation is exact since

$$\frac{\partial}{\partial y}(x^3 + 3xy^2) = \frac{\partial}{\partial x}(3x^2y + y^3) = 6xy.$$

It is clear that

$$\int (x^3 + 3xy^2)\, dx = \frac{x^4}{4} + \frac{3x^2y^2}{2} \text{ and } \int (3x^2y + y^3)\, dy = \frac{3x^2y^2}{2} + \frac{y^4}{4}$$

and hence, the general solution is $x^4 + 6x^2y^2 + y^4 = c$. □

Definition 1.10 (integrating factor). A non-zero function $\mu(x,y)$ is an integrating factor of $M(x,y)dx + N(x,y)dy = 0$ if the equivalent differential equation

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

is exact.

If μ is an integrating factor, then $(\mu M)_y = (\mu N)_x$, so by the product rule,

$$\mu M_y + \mu_y M = \mu N_x + \mu_x N \quad \text{so} \quad N\mu_x - M\mu_y = \mu(M_y - N_x).$$

One may look for an integrating factor of the form $\mu = \mu(v)$, where v is a known function of x and y. Hence,

$$\frac{1}{\mu}\frac{d\mu}{dv} = \frac{M_y - N_x}{Nv_x - Mv_y}.$$

If the right side is a function of v alone, say $\phi(v)$, then

$$\int \frac{1}{\mu}\, d\mu = \int \phi(v)\, dv$$
$$\mu = \exp\left(\int \phi(v)\, dv\right)$$

We use this method of integrating factor to help us solve differential equations. There are three common choices for v, namely $v = x$, $v = y$ and $v = xy$.

Example 1.9. Solve the differential equation

$$x^2y + y + 1 + x(1+x^2)\frac{dy}{dx} = 0.$$

Solution. Note that $M(x,y) = x^2y + y + 1$ and $N(x,y) = x(1+x^2)$. Let μ be an integrating factor of the differential equation. Then, $M_y - N_x = -2x^2$. It would be apt if $Nv_x - Mv_y$ is a function solely in terms of x, which prompts us to set $v = x$. Hence, $Nv_x - Mv_y = x(1+x^2)$, implying that

$$\mu = \frac{1}{1+x^2}.$$

Multiplying both sides of the differential equation by μ gives us

$$y + \frac{1}{1+x^2} + xy' = 0,$$

and we observe that it is an exact equation. Setting $u_x = y + \frac{1}{1+x^2}$ and $u_y = x$, integrating the first equation gives $u(x,y) = xy + \tan^{-1} x + \phi(y)$. Taking the partial derivative with respect to y, we have $u_y = x + \phi'(y)$ but since $u_y = x$ as mentioned, it implies that $\phi'(y) = 0$, and so $\phi(y) = c$. Therefore, the general solution is

$$xy + \tan^{-1} x = c.$$

□

Example 1.10. Solve the differential equation

$$xy^3 + 2x^2y^2 - y^2 + (x^2y^2 + 2x^3y - 2x^2)\frac{dy}{dx} = 0.$$

Solution. Consider $M_y - N_x = xy^2 - 2x^2y + 4x - 2y$. It hints to us that $Nv_x - Mv_y$ should be a function in terms of xy. Consider the substitution $v = xy$. Then, $Ny - Mx = -x^2y$. Putting everything together,

$$\frac{M_y - N_x}{Ny - Mx} = 1 - \frac{2}{xy},$$

which is a function in terms of xy. The integrating factor is

$$\mu = \exp\left(\int 1 - \frac{2}{v}\, dv\right) = \frac{e^{xy}}{(xy)^2}.$$

Multiplying both sides of the differential equation by μ gives

$$e^{xy}\left(\frac{y}{x} + 2 - \frac{1}{x^2}\right) + e^{xy}\left(1 + \frac{2x}{y} - \frac{2}{y^2}\right)\frac{dy}{dx} = 0,$$

which is an exact equation. Taking the integral of M with respect to x and using integration by parts,

$$\begin{aligned}
u(x,y) &= \int e^{xy}\left(yx^{-1}+2-x^{-2}\right)\,dx \\
&= y\int e^{xy}x^{-1}\,dx + 2\int e^{xy}\,dx - \int e^{xy}x^{-2}\,dx + \phi(y) \\
&= y\int e^{xy}x^{-1}\,dx + 2e^{xy}y^{-1} - \int e^{xy}x^{-2}\,dx + \phi(y) \\
&= y\int e^{xy}x^{-1}\,dx + 2e^{xy}y^{-1} + e^{xy}x^{-1} + y\int e^{xy}x^{-1}\,dx + \phi(y) \\
&= e^{xy}\left(2y^{-1}+x^{-1}\right) + \phi(y)
\end{aligned}$$

As $u_y = e^{xy}(1+2xy^{-1}-2y^{-2})$, then taking the partial derivative of $u(x,y)$ with respect to y, we have $u_y = e^{xy}(1+2xy^{-1}-2y^{-2}+\phi'(y)$. Set $\phi'(y)=0$, so $\phi(y)=c$. Hence, the solution to the differential equation is

$$e^{xy}(x^{-1}+2y^{-1}) = c,$$

where c is a constant. □

Proposition 1.2. Some useful formulae are as follows:

(1)
$$d\left(\frac{x}{y}\right) = \frac{ydx-xdy}{y^2}$$

(2)
$$d(xy) = xdy + ydx$$

(3)
$$d\left(x^2+y^2\right) = 2xdx + 2ydy$$

(4)
$$d\left(\tan^{-1}\left(\frac{x}{y}\right)\right) = \frac{ydx-xdy}{x^2+y^2}$$

(5)
$$d\left(\ln\left|\frac{x}{y}\right|\right) = \frac{ydx-xdy}{xy}$$

1.3. *First-Order Linear Equations*

Definition 1.11 (homogeneous linear equation). A first order homogeneous linear equation is of the form

$$y' + yP(x) = 0,$$

where $P(x)$ is a continuous function on an interval J.

Theorem 1.2. The general solution to the differential equation

$$y' + yP(x) = 0 \quad \text{is} \quad y = ce^{-P(x)} \text{ where } P(x) = \int_a^x P(s)\,ds.$$

Proof. Set $P(x) = \int_a^x P(s)\,ds$. Then, multiplying both sides of the equation by $e^{P(x)}$, we obtain

$$\frac{d}{dx}\left(ye^{P(x)}\right) = 0,$$

which implies that $ye^{P(x)}$ is some constant, say c. The result follows. □

Definition 1.12 (homogeneous non-linear equation). We consider a first order non-homogeneous linear equation, namely

$$y' + yP(x) = Q(x),$$

where $P(x)$ and $Q(x)$ are continuous functions on an interval J.

Theorem 1.3. The general solution to the differential equation

$$y' + yP(x) = Q(x) \quad \text{is} \quad y = e^{-P(x)} \int_a^x e^{P(t)} Q(t)\, dt \text{ where } P(x) = \int_a^x P(s)\, ds$$

Proof. Set $P(x) = \int_a^x P(s)\, ds$. Multiplying the original differential equation by $e^{P(x)}$ gives

$$\frac{d}{dx}\left(ye^{P(x)}\right) = e^{P(x)}Q(x).$$

Integrating both sides with respect to x and dividing by $e^{P(x)}$, we obtain the general solution. □

Example 1.11. Solve the differential equation

$$x\frac{dy}{dx} + 3y = 5x^2, \text{ where } x > 0.$$

Solution: We divide both sides by x so that the differential equation becomes

$$\frac{dy}{dx} + 3\left(\frac{y}{x}\right) = 5x,$$

thus it is clear that $P(x) = 3/x$ and $Q(x) = 5x$. The integrating factor is $\exp\left(\int 3/x\right) = x^3$. Multiplying both sides by the integrating factor, it is easy to see that

$$\frac{d}{dx}\left(x^3 y\right) = 5x^4.$$

Integrating both sides with respect to x and dividing by x^3, we conclude that the solution is

$$y = x^2 + \frac{c}{x^3}.$$

□

Definition 1.13 (Bernoulli equation). An ODE of the form

$$y' + yP(x) = y^n Q(x),$$

where $n \neq 0, 1$, is called the Bernoulli equation. The functions $P(x)$ and $Q(x)$ are continuous functions on an interval J.

To solve differential equations of this form, we need to perform a trick such that it reduces to a non-homogeneous linear equation (discussed in the previous section). Consider the substitution $u = y^{1-n}$. Then,

$$\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx} \quad \text{so} \quad \frac{y^n}{1-n}\left(\frac{du}{dx}\right) = \frac{dy}{dx}.$$

Hence, the differential equation becomes

$$\frac{du}{dx} + (1-n)uP(x) = (1-n)Q(x),$$

which is indeed a first order linear ODE.

Definition 1.14 (Riccati equation). An ODE of the form

$$y' = P(x) + yQ(x) + y^2 R(x)$$

is called a Riccati Equation. The functions $P(x), Q(x)$ and $R(x)$ are continuous on an interval J.

In general, the Riccati equation cannot be solved by a sequence of integrations. However, if a particular solution is known, then it can be reudced to a linear equation, and thus is solvable. The constraints on the Riccati equation are that $P(x) \neq 0$ and $R(x) \neq 0$. Note that if $P(x) = 0$, we obtain the Bernoulli equation and if $R(x) = 0$, we obtain a first order non-homogeneous linear equation which can be easily solved by the method of integrating factor.

Let us come up with an approach to solve the Riccati equation.

Theorem 1.4. If $y = y_0(x)$ is a particular solution of the Riccati equation, we set $H(x)$ and $Z(x)$ to be the following:

$$H(x) = \int_{x_0}^{x} Q(t) + 2R(t)y_0(t)\, dt$$

$$Z(x) = e^{-H(x)} \left(c - \int_{x_0}^{x} e^{H(t)} R(t)\, dt \right)$$

where c is an arbitrary constant. Then, the general solution is given by

$$y = y_0(x) + \frac{1}{Z(x)}.$$

Proof. Let $u(x)z(x) = 1$ and $y = y_0(x) + u(x)$, which yields

$$y' = y_0' + u' = P + Q(y_0 + u) + R(y_0 + u)^2.$$

As $y = y_0$ is a particular solution, then $y_0' = P + Qy_0 + Ry_0^2$, which implies that $u' = (Q + 2Ry_0)u + Ru^2$. This is a Bernoulli equation, where $n = 2$. The rest of the proof is left as an exercise since I believe you know where I'm heading towards. □

Example 1.12. Solve the Riccati equation

$$\frac{dy}{dx} = -x^5 + \frac{y}{x} + x^3 y^2,$$

where $y_p = x$ is a particular solution.

Solution. Let the general solution be $y = x + u$. Differentiating both sides yields $y' = 1 + u'$, and so

$$\frac{du}{dx} = -x^5 + \frac{u}{x} + x^3 (x + u)^2.$$

Using the substitution $uz = 1$, we have $uz' + zu' = 0$ and so the differential equation becomes

$$-\frac{u}{z}\left(\frac{dz}{dx}\right) = -x^5 + \frac{1}{xz} + x^3 \left(\frac{xz+1}{z}\right)^2$$

$$\frac{dz}{dx} = \frac{x^5 z}{u} - \frac{1}{ux} + \frac{x^3 (xz+1)^2}{uz}$$

$$\frac{dz}{dx} = x^5 z^2 - \frac{z}{x} + x^3 (xz+1)^2$$

$$\frac{dz}{dx} + z\left(\frac{1}{x} + 2x^4\right) = -x^3$$

The rest of the working uses the method of integrating factor so I shall not delve into it. The general solution is given by

$$y = x + \frac{x\exp\left(\frac{2x^5}{5}\right)}{c - \frac{1}{2}\exp\left(\frac{2x^5}{5}\right)},$$

for a constant c. □

Example 1.13 (MA3220 AY24/25 Sem 1 Midterm)**.** Given that $y_0 = x$ is a particular solution to the Riccati equation

$$y' = \left(1 + x + 2x^2\cos x\right) - (1 + 4x\cos x)\,y + 2y^2\cos x,$$

find its general solution.

Solution. Let the general solution be $y = y_0 + u = x + u$. Then,

$$\frac{dy}{dx} = 1 + \frac{du}{dx}.$$

As such,

$$\begin{aligned}
1 + \frac{du}{dx} &= 1 + x + 2x^2\cos x - (x+u)(1 + 4x\cos x) + 2(x+u)^2\cos x \\
\frac{du}{dx} &= x + 2x^2\cos x - x - u - 4x^2\cos x - 4ux\cos x + 2x^2\cos x + 4ux\cos x + 2u^2\cos x \\
\frac{du}{dx} &= -u + 2u^2\cos x \\
\frac{1}{u^2}\frac{du}{dx} &= -\frac{1}{u} + 2\cos x
\end{aligned}$$

By using the substitution $v = 1/u$, we obtain

$$\frac{dv}{dx} - v = -2\cos x.$$

The integrating factor is e^{-x} so

$$\frac{d}{dx}\left(ve^{-x}\right) = -2e^{-x}\cos x.$$

Using integration by parts,

$$v = \frac{1}{e^{-x}}\left(e^{-x}\cos x - e^{-x}\sin x + c\right) \quad \text{so} \quad v = \cos x - \sin x + ce^x.$$

The general solution is

$$y = x + \frac{1}{\cos x - \sin x + ce^x}.$$

□

Theorem 1.5 (Riccati equation)**.** The general solution of the Riccati equation can be written as

$$y = \frac{cF(x) + G(x)}{cf(x) + g(x)},$$

where

$$\begin{aligned} f(x) &= e^{-H(x)} \\ g(x) &= -f(x)\int_{x_0}^{x} e^{H(t)}R(t)\ dt \\ F(x) &= y_0(x) f(x) \\ G(x) &= y_0(x) g(x) + 1 \end{aligned}$$

Definition 1.15 (cross ratio). Define the cross ratio of four distinct functions $p(x), q(x), r(x)$ and $s(x)$ by

$$\frac{(p-q)(r-s)}{(p-s)(r-q)}.$$

The cross ratio of four distinct particular solutions of a Riccati equation is independent of x. As a consequence, we have the following result (Theorem 1.6):

Theorem 1.6. If y_1, y_2 and y_3 are three distinct particular solutions of a Riccati equation, then the general solution is given by

$$\frac{(y_1-y_2)(y_3-y)}{(y_1-y)(y_3-y_2)} = c \quad \text{where } c \text{ is a constant.}$$

Theorem 1.7. If y_1 and y_2 are two distinct particular solutions of a Riccati equation, its general solution is

$$\ln\left|\frac{y-y_1}{y-y_2}\right| = \int (y_1(x) - y_2(x))R(x)\ dx + c \quad \text{where } c \text{ is a constant.}$$

Definition 1.16 (Abel's equation). A generalisation of the Riccati equation is Abel's equation of the first kind. The latter has the formula

$$y' = P(x) + yQ(x) + y^2R(x) + y^3S(x),$$

where $P(x), Q(x), R(x)$ and $S(x)$ are continuous functions on an interval J and $S(x) \neq 0$.

Abel's equation can be reduced to either a Riccati equation or a Bernoulli equation.

1.4. *First-Order Implicit Equations*

Previously, we discussed first-order explicit equations, i.e. equations of the form $y' = f(x,y)$. Now, we will discuss solutions of some first-order explicit equations

$$F\left(x,y,y'\right) = 0 \quad \text{which are not sovlable in } y'.$$

Consider an equation solvable in y, say $y = f(x,y')$. Let $p = y'$. Differentiating $y = f(x,p)$, we obtain

$$\left(f_x(x,p) - p\right)\ dx + f_p(x,p)\ dp = 0.$$

This is a first-order explicit equation in terms of x and p. If $p = \phi(x)$ is a solution of the above equation, then $y = f(x,\phi(x))$ is a solution to $y = f(x,y')$.

This prompts us to discuss Clairaut's equation (Definition 1.17).

Definition 1.17 (Clairaut's equation)**.** Clairaut's euqation is of the form

$$y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right) \quad \text{where } f \text{ has continuous first-order derivative.}$$

The trick to solving Clairaut's equation is to let $p = y'$.

Example 1.14. Solve Clairaut's equation

$$y = x\frac{dy}{dx} - \frac{1}{4}\left(\frac{dy}{dx}\right)^2.$$

Solution. Let $p = dy/dx$. Then, differentiating both sides of the original equation yields

$$\frac{dy}{dx} = x\frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{1}{2}\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right).$$

As such,

$$\begin{aligned} p &= x\frac{dp}{dx} + p - \frac{1}{2}p\frac{dp}{dx} \\ \frac{dp}{dx}(2x - p) &= 0 \end{aligned}$$

Either $dp/dx = 0$ or $p = 2x$. The rest of the working is trivial. One should be able to deduce that $y = cx - c^2/4$ or $y = x^2$ respectively. □

We then discuss the method of parametrisation, which can be used to solve equations where either x or y is missing. Consider $F(y, y') = 0$, where x is missing. Let $p = y'$ and we can write the differential equation as $F(y, p) = 0$. This determines a family of curves in the yp-plane. Let one of the curves be defined by $y = g(t)$ and $p = h(t)$ parametrically, i.e. $F(g(t), h(t)) = 0$. Since

$$y' = \frac{dy}{dx} \quad \text{then} \quad dx = \frac{dy}{y'} = \frac{dy}{p} = \frac{g'(t)\ dt}{h(t)},$$

and consequently, we have

$$x = \int_{t_0}^{t} \frac{g'(t)}{h(t)}\, dt + c.$$

The solutions to the original differential equation $F(y, y')$ are thus

$$x = \int_{t_0}^{t} \frac{g'(t)}{h(t)}\, dt + c \quad \text{and} \quad y = g(t).$$

Example 1.15. Solve

$$y^2 + \left(\frac{dy}{dx}\right)^2 - 1 = 0.$$

Solution. We can use the method of parametrisation. Alternatively, we will use a different method. Differentiating the original differential equation once, we obtain

$$\begin{aligned} 2y\frac{dy}{dx} + 2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) &= 0 \\ \frac{dy}{dx}\left(y + \frac{d^2y}{dx^2}\right) &= 0 \end{aligned}$$

Either $dy/dx = 0$ or $y + d^2y/dx^2 = 0$. We omit the remaining details (solving the second-order ODE requires a technique on finding its characteristic equation which we will discuss in due course). □

Thirdly, we also have the method of reduction of order. Consider the equation $F(x, y', y'') = 0$, where y is missing. Let $p = y'$, then $y'' = p'$. As such, the differential equation can be written as $F(x, p, p') = 0$. This is a first-order equation in x and p. If $p = \phi(x, c_1)$ is a general solution to $F(x, p, p') = 0$, then the general solution to $F(x, y', y'') = 0$ is

$$y = \int_{x_0}^{x} \phi(t, c_1)\ dt + c_2.$$

Example 1.16. Solve the differential equation

$$x \frac{d^2 y}{dx^2} - \frac{dy}{dx} = 3x^2.$$

Solution. We use the substitution $p = dy/dx$ so the differential equation becomes

$$x \frac{dp}{dx} - p = 3x^2.$$

As such, we now use the method of integrating factor. One should check that the solution is $y = x^3 + c_1 x^2 + c_2$ for some constants c_1 and c_2. □

2. Linear Differential Equations

2.1. *General Theory*

Consider the n^{th} order linear equation

$$y^{(n)} + a_1(x)\,y^{(n-1)} + \ldots + a_{n-1}(x)\,y' + a_n(x)\,y = f(x),$$

where $y^{(k)}$ denotes the k^{th} derivative of y with respect to x. Throughout this chapter, we assume that $a_j(x)$ and $f(x)$ are continuous functions defined on the interval (a,b). When $f(x) \neq 0$, the above equation is said to be non-homogeneous. As such, the associated homogeneous equation is

$$y^{(n)} + a_1(x)\,y^{(n-1)} + \ldots + a_{n-1}(x)\,y' + a_n(x)\,y = 0.$$

We begin with the initial value problem

$$\begin{aligned} y^{(n)} + a_1(x)\,y^{(n-1)} + \ldots + a_n(x)\,y &= f(x) \\ y(x_0) &= y_0 \\ y'(x_0) &= y_1 \\ \vdots &= \vdots \\ y^{(n-1)}(x_0) &= y_{n-1} \end{aligned}$$

Theorem 2.1 (existence and uniqueness theorem)**.** Assume that $a_1(x),\ldots,a_n(x)$ as well as $f(x)$ are continuous functions defined on the interval (a,b). Then, for any $x_0 \in (a,b)$ and for any numbers $y_0,\ldots,y_{n-1}$, the initial value problem

$$\begin{aligned} y^{(n)} + a_1(x)\,y^{(n-1)} + \ldots + a_n(x)\,y &= f(x) \\ y(x_0) &= y_0 \\ y'(x_0) &= y_1 \\ \vdots &= \vdots \\ y^{(n-1)}(x_0) &= y_{n-1} \end{aligned}$$

has a unique solution defined on (a,b). Especially if $a_j(x)$ and $f(x)$ are continuous on $\mathbb{R}$, then for any x_0 and $y_0,\ldots,y_{n-1}$, the initial value problem has a unique solution defined on $\mathbb{R}$.

Corollary 2.1. Let $y = y(x)$ be a solution to the homogeneous equation

$$y^{(n)} + a_1(x)\,y^{(n-1)} + \ldots + a_{n-1}(x)\,y' + a_n(x)\,y = 0 \quad \text{in an interval } (a,b).$$

Assume that there exists $x_0 \in (a,b)$ such that

$$y(x_0) = 0 \quad y'(x_0) = 0 \quad \ldots \quad y^{(n-1)}(x_0) = 0.$$

Then, $y(x) = 0$ on (a,b).

We consider general solutions to both the homogeneous and non-homogeneous cases. Given continuous functions $a_j(x)$, where $0 \le j \le n$ and $f(x)$, define an operator L as follows:

$$L[y] = a_0(x)\,y^{(n)} + a_1(x)\,y^{(n-1)} + \ldots + a_n(x)\,y$$

Then, the following properties hold:

Proposition 2.1. We have the following:

(i) $L[cy] = cL[y]$ for any constant c

(ii) $L[u+v] = L[u] + L[v]$

An operator satisfying these properties is said to be a linear operator.

Example 2.1. The differential operator L is a linear operator.

Note that the homogeneous and non-homogeneous equations discussed at the start of this chapter can be written as

$$L[y] = 0 \text{ and } L[y] = f(x) \quad \text{respectively.}$$

As such, we obtain the following:

Theorem 2.2. The following hold:

(i) **Superposition principle:** If y_1 and y_2 are solutions of the homogeneous equation in an interval (a,b), then for any constants c_1 and c_2,

$$y = c_1 y_1 + c_2 y_2 \quad \text{is also a solution to the homogeneous equation on } (a,b).$$

(ii) If y_p is a solution to the non-homogeneous equation (called a particular solution) and y_h is a solution to the homogeneous equation on (a,b), then

$$y = y_h + y_p \quad \text{is also a solution to the non-homogeneous equation on } (a,b).$$

In order to discuss the structure of solutions, we need to introduce the idea of linear independence.

Definition 2.1 (linear independence). Functions $\phi_1(x), \ldots, \phi_k(x)$ are linearly dependent on (a,b) if there exist constants $c_1, \ldots, c_k$, not all zero, such that

$$c_1\phi_1(x) + \ldots + c_k\phi_k(x) = 0 \quad \text{for all } x \in (a,b).$$

A set of functions is linearly independent on (a,b) if they are not linearly dependent on (a,b).

Lemma 2.1. Functions $\phi_1(x), \ldots, \phi_k(x)$ are linearly dependent on (a,b) if and only if the following vector values functions

$$\begin{bmatrix} \phi_1 \\ \phi_1' \\ \vdots \\ \phi_1^{(n-1)} \end{bmatrix}, \ldots, \begin{bmatrix} \phi_k \\ \phi_k' \\ \vdots \\ \phi_k^{(n-1)} \end{bmatrix} \quad \text{are linearly dependent on } (a,b).$$

Definition 2.2 (Wronskian). Let $\phi_1(x), \ldots, \phi_n(x)$ be n functions. We define their Wronskian W to be

$$W(\phi_1, \ldots, \phi_n)(x) = \det \begin{bmatrix} \phi_1 & \ldots & \phi_n \\ \vdots & \ddots & \vdots \\ \phi_1^{(n-1)} & \ldots & \phi_n^{(n-1)} \end{bmatrix}$$

Theorem 2.3. Let $y_1(x), \ldots, y_n(x)$ be n solutions of

$$y^{(n)} + a_1(x) y^{(n-1)} + \ldots + a_{n-1}(x) y' + a_n(x) y = 0$$

on (a,b) and W be their Wronskian.

(i) $y_1(x), \ldots, y_n(x)$ are linearly dependent on (a,b) if and only if $W(x) = 0$ on (a,b)

(ii) $y_1(x), \ldots, y_n(x)$ are linearly independent on (a,b) if and only if $W(X)$ does not vanish on (a,b)

Corollary 2.2. The Wronskian of n solutions of

$$y^{(n)} + a_1(x) y^{(n-1)} + \ldots + a_{n-1}(x) y' + a_n(x) y = 0$$

is either identically zero, or nowhere zero. Also, n solutions $y_1, \ldots, y_n$ of

$$y^{(n)} + a_1(x) y^{(n-1)} + \ldots + a_{n-1}(x) y' + a_n(x) y = 0$$

are linearly independent on (a,b) if and only if the set of vectors

$$\begin{bmatrix} y_1(x_0) \\ y_1'(x_0) \\ \vdots \\ y_1^{(n-1)}(x_0) \end{bmatrix}, \ldots, \begin{bmatrix} y_n(x_0) \\ y_n'(x_0) \\ \vdots \\ y_n^{(n-1)}(x_0) \end{bmatrix}$$

are linearly independent for some $x_0 \in (a,b)$.

Example 2.2. Consider the differential equation

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} = 0 \quad \text{for } x > 0.$$

Both $\phi_1(x) = 1$ and $\phi_2(x) = x^2$ are solutions of the differential equation. Also,

$$W(\phi_1, \phi_2)(x) = \det \begin{bmatrix} 1 & x^2 \\ 0 & 2x \end{bmatrix} = 2x \neq 0 \quad \text{as } x > 0.$$

Thus, ϕ_1 and ϕ_2 are linearly independent solutions.

Theorem 2.4. We have the following:

(i) Let $a_1(x), \ldots, a_n(x)$ and $f(x)$ be continuous on the interval (a,b). The homogeneous equation

$$y^{(n)} + a_1(x) y^{(n-1)} + \ldots + a_{n-1}(x) y' + a_n(x) y = 0$$

has n linearly independent solutions on (a,b).

(ii) Let $y_1, \ldots, y_n$ be n linearly independent solutions of

$$y^{(n)} + a_1(x) y^{(n-1)} + \ldots + a_{n-1}(x) y' + a_n(x) y = 0$$

defined on (a,b). The general solution to this differential equation is

$$y(x) = c_1 y_1(x) + \ldots + c_n y_n(x) \quad \text{for some constants } c_1, \ldots, c_n.$$

2.2. *Linear Equations with Constant Coefficients*

We begin with second-order linear equations with constant coefficients, i..e

$$\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by=0.$$

Here, a and b are constants. We look for a solution of the form $y=e^{\lambda x}$. Substituting this into the differential equation, we see that

$$e^{\lambda x} \text{ is a solution to } y''+ay'+by=0 \quad \text{if and only if} \quad \lambda^2+a\lambda+b=0.$$

This equations is called the auxiliary equation or the characteristic equation of the differential equation. The roots of the characteristic equation are called characteristic values, or eigenvalues. As such,

$$\lambda_1=\frac{-a+\sqrt{a^2-4b}}{2} \quad \text{and} \quad \lambda_2=\frac{-a-\sqrt{a^2-4b}}{2}.$$

Theorem 2.5. Based on our earlier discussion, we have three cases to consider.

(i) If $a^2-4b>0$, then the characteristic equation has two real and distinct roots λ_1 and λ_2, and the general solution to the differential equation is

$$y=c_1e^{\lambda_1x}+c_2e^{\lambda_2x}.$$

(ii) If $a^2-4b=0$, then the characteristic equation has only one real root λ, i.e. $\lambda_1=\lambda_2$. The general solution to the differential equation is

$$y=c_1e^{\lambda_1x}+c_2xe^{\lambda_2x}.$$

(iii) If $a^2-4b<0$, then the characteristic equation has a pair of complex conjugate roots

$$\lambda_1=\alpha+\beta_i \quad \text{and} \quad \lambda_2=\alpha-\beta i.$$

So, the general solution to the differential equation is

$$y=c_1e^{\alpha x}\cos\beta x+c_2e^{\alpha x}\sin\beta x.$$

Example 2.3. Solve

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=0 \quad \text{with initial conditions } y(0)=4 \text{ and } y'(0)=-5.$$

Solution. Check that the roots of the characteristic equation are $\lambda_1=1$ and $\lambda_2=-2$ so the solution to the differential equation is $y=e^x+3e^{-2x}$. □

Example 2.4. Solve the differential equation

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0 \quad \text{with initial conditions } y(0)=3 \text{ and } y'(0)=1.$$

Solution. The roots of the characteristic equation are repeated, i.e. $\lambda_1=\lambda_2=2$. One checks that the solution to the differential equation is $y=(3-5x)e^{2x}$. □

Example 2.5. Solve the differential equation

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+10y=0.$$

Solution. The roots of the characteristic equation are $\lambda_1 = 1+3i$ and $\lambda_2 = 1-3i$. As such, it is clear that the general solution to the differential equation is $y = e^x (c_1 \cos 3x + c_2 \sin 3x)$. □

Now, we consider n^{th} order homogeneous linear equations with constant coefficients, i.e.

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = 0,$$

where $a_1, \ldots, a_n$ are real constants. Again, $y = e^{\lambda x}$ is a solution to the differential equation if and only if λ satisfies the characteristic equation

$$\lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0.$$

Let $\lambda_1, \ldots, \lambda_n$ be the distinct eigenvalues of the characteristic equation. Then, we can write

$$\lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \ldots (\lambda - \lambda_s)^{m_s}$$

where $m_1, \ldots, m_s \in \mathbb{N}$ and $m_1 + \ldots + m_s = n$. We call the m_i's the multiplicity of the eigenvalues λ_i respectively.

Lemma 2.2. Assume that λ is an eigenvalue of the differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = 0 \quad \text{of multiplicity } m.$$

Then, the following hold:

(i) $e^{\lambda x}$ is a solution to the differential equation

(ii) If $m > 1$, then for any positive integer $1 \le k \le m-1$, $x^k e^{\lambda x}$ is a solution to the differential equation

(iii) If $\lambda = \alpha + \beta i$, then

$$x^k e^{\alpha x} \cos \beta x \text{ and } x^k e^{\alpha x} \sin \beta x \quad \text{are solutions of the differential equation, where } 0 \le k \le m-1$$

We then discuss solutions to the non-homogeneous equation

$$\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x) y = f(x).$$

The associated homogeneous equation is

$$\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x) y = 0,$$

for which this method also applies to higher-order equations.

First, we have the variation of parameters. Let y_1 and y_2 be two linearly independent solutions of the associated homogeneous equation

$$\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x) y = 0$$

and let $W(x)$ be their Wronskian. We look for a particular solution to the non-homogeneous equation of the form

$$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$$

where u_1 and u_2 are functions to be determined. Suppose

$$u_1' y_1 + u_2' y_2 = 0.$$

Differentiating this equation once, we obtain

$$u_1'' y_1 + u_2'' y_2 = -u_1' y_1' - u_2' y_2'.$$

As such, we obtain

$$u_1' y_1' + u_2' y_2' = f$$

which implies u_1' and u_2' satisfy

$$u_1' y_1 + u_2' y_2 = 0 \quad \text{and} \quad u_1' y_1' + u_2' y_2' = f.$$

Solving this system, we obtain

$$u_1' = -\frac{y_2}{W} f \quad \text{and} \quad u_2' = \frac{y_1}{W} f.$$

Integrating both yield

$$u_1(x) = -\int_{x_0}^{x} \frac{y_2(t)}{W(t)} f(t)\, dt \quad \text{and} \quad u_2(x) = \int_{x_0}^{x} \frac{y_1(t)}{W(t)} f(t)\, dt.$$

Example 2.6. Solve the differential equation

$$\frac{d^2 y}{dx^2} + y = \sec x.$$

Solution. A basis for the solutions of the homogeneous equation consists of $y_1 = \cos x$ and $y_2 = \sin x$. Note that the Wronskian $W(y_1, y_2) = 1$ so

$$u_1 = -\int \sin x \sec x\, dx = \ln|\cos x| + c_1$$
$$u_2 = \int \cos x \sec x\, dx = x + c_2$$

From this, a particular solution is obtained by

$$y_p = \cos x \ln|\cos x| + x \sin x$$

so the general solution is

$$y = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x.$$

□

The method of variation of parameters can also be used to find another solution of a second-order homogeneous linear differential equation when one solution is given. Suppose z is a known solution to the equation

$$\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x) y = 0.$$

Assume that $y = vz$ is a solution so one can deduce that

$$\frac{v''}{v'} = -2\left(\frac{z'}{z}\right) - P.$$

As such,

$$v' = \frac{1}{z^2} e^{-\int P\, dx} \quad \text{so} \quad v = \int \frac{1}{z^2} e^{-\int P\, dx}.$$

One can show that z and vz are linearly independent solutions by computing their Wronskian.

Example 2.7. Given that $y_1 = x$ is a solution to

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0,$$

find another solution.

Solution. We shall write the differential equation as

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y=0.$$

Then, $P(x)=1/x$. Assume that $y=vx$ is another linearly independent solution. Differentiating both sides yields

$$\frac{dy}{dx}=v+x\frac{dv}{dx} \quad \text{so} \quad \frac{d^2y}{dx^2}=2\frac{dv}{dx}+x\frac{d^2v}{dx^2}$$

Hence,

$$2\frac{dv}{dx}+x\frac{d^2v}{dx^2}+\frac{1}{x}\left(v+x\frac{dv}{dx}\right)-\frac{vx}{x^2}=0$$
$$x\frac{d^2v}{dx^2}+3\frac{dv}{dx}=0$$

We see that we deviated from the method that was suggested but anyway, this new differential equation is known as a Cauchy-Euler equation which we will mention in due course. We will not discuss the method here (fairly straightforward just like finding the characteristic equation of the corresponding differential equation) but anyway,

$$v=-\frac{1}{2x^2} \quad \text{so} \quad y=-\frac{1}{2x}.$$

As such, the general solution is $y=c_1x+c_2/x$. □

Next, we also have the method of undetermined coefficients. Consider the equation

$$\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by=f(x),$$

where a and b are real constants.

- **Case 1:** Suppose $f(x)=P_n(x)e^{\alpha x}$, where $P_n(x)$ is a polynomial of degree $n\geq 0$. We look for a particular solution of the form $y=Q(x)e^{\alpha x}$, where $Q(x)$ is a polynomial. Substituting this into the original differential equation, we obtain

$$Q''+(2\alpha+a)Q'+\left(\alpha^2+a\alpha+b\right)Q=P_n(x).$$

  - **Subcase 1:** Suppose $\alpha^2+a\alpha+b\neq 0$, i.e. α is not a root of the characteristic equation. Then, we can choose $Q=R_n$, which is a polynomial of degree n, and $y=R_n(x)e^{\alpha x}$. The coefficients of R_n can be determined by comparing the terms of the same power on the two sides of the above equation.
  - **Subcase 2:** If $\alpha^2+a\alpha+b=0$ but $2\alpha+a\neq 0$, i.e. α is a simple root (of multiplicity 1) of the characteristic equation, then the equation above is reduced to

$$Q''+(2\alpha+a)Q'=P_n.$$

  We choose Q to be a polynomial of degree $n+1$. Since the constant term of Q does not appear in $Q''+(2\alpha+a)Q'=P_n$, we can choose $Q(x)=xR_n(x)$, where $R_n(x)$ is a polynomial of degree n. As such, $y=xR_n(x)e^{\alpha x}$.

- **Subcase 3:** If $\alpha^2 + a\alpha + b = 0$ and $2\alpha + a = 0$, then α is a root of the characteristic equation of multiplicity 2. As such, we obtain $Q'' = P_n$ so we choose $Q(x) = x^2 R_n(x)$, where $R_n(x)$ is a polynomial of degree n. As such, $y = x^2 R_n(x) e^{\alpha x}$.

We first deal with some examples.

Example 2.8. Find the general solution to the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 4x^2.$$

Solution. The solution to the homogeneous equation is $y_h = c_1 e^{2x} + c_2 e^{-x}$. We then try to deduce the particular solution, for which we infer that y should be a quadratic polynomial $ax^2 + bx + c$. Substituting this into the original differential equation, one should deduce that $y_p = -2x^2 + 2x - 3$. □

Example 2.9. Find a particular solution to the differential equation

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - \frac{dy}{dx} = 3x^2 - 2x + 1.$$

Solution. Similar to Example 2.8, we infer that the particular solution y_p should be some cubic polynomial $ax^3 + bx^2 + cx + d$. Substituting this into the differential equation, one should be able to obtain that $y_p = -x^3 - 5x^2 - 27x$. □

Example 2.10. Solve the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x.$$

Solution. The solution to the homogeneous equation is $y_h = c_1 e^x + c_2 x e^x$. One checks that the particular solution is of the form $y_p = cx^2 e^x$. However, this fails so we try $y_p = cx^3 e^x$. This works and in fact, one can deduce that $c = 1/6$. □

We return to our casework.

- **Case 2:** Suppose

$$f(x) = P_n(x) e^{\alpha x} \cos \beta x \quad \text{or} \quad f(x) = P_n(x) e^{\alpha x} \sin \beta x,$$

where $P_n(x)$ is a polynomial of degree $n \geq 0$. We first look for a solution to the differential equation

$$y'' + ay' + b' = P_n(x) e^{(\alpha + \beta i)x}.$$

We obtain a complex-valued solution where $u(x) = \operatorname{Re} z$ and $v(x) = \operatorname{Im} z$. Substituting $z(x) = u(x) + v(x)$ into $y'' + ay' + b' = P_n(x) e^{(\alpha+\beta i)x}$ and taking the real and imaginary parts, we deduce that $u(x) = \operatorname{Re} z$ and $v(x) = \operatorname{Im} z$ are solutions to

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = P_n(x) e^{\alpha x} \cos \beta x \quad \text{and} \quad \frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = P_n(x) e^{\alpha x} \sin \beta x \quad \text{respectively.}$$

Example 2.11. Solve the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = e^x \cos x.$$

Solution. The solution to the homogeneous equation is $y_h = c_1 e^x \cos x + c_2 e^x \sin x$. Suppose the particular solution is of the form

$$y_p = P(x) e^x (A \cos x + B \sin x).$$

Substituting this into the original differential equation, one can deduce that the particular solution is $\frac{1}{2} x e^x \sin x$. Combining the homogeneous solution y_h and the particular solution y_p, we obtain the general solution. □

Theorem 2.6. Let y_1 and y_2 be particular solutions of the equations

$$y'' + ay' + by = f_1(x) \quad \text{and} \quad y'' + ay' + by = f_2(x) \quad \text{respectively.}$$

Then, $y_p = y_1 + y_2$ is a particular solution of

$$y'' + ay' + by = f_1(x) + f_2(x).$$

Example 2.12. Solve the differential equation

$$\frac{d^2y}{dx^2} - y = e^x + \sin x.$$

Solution. A particular solution for $y'' - y = e^x$ is given by $y_1 = \frac{1}{2}xe^x$. Also, a particular solution for $y'' - y = \sin x$ is given by $y_2 = -\frac{1}{2}\sin x$. As such, we obtain the particular solution $y_p = \frac{1}{2}(xe^x - \sin x)$. The homogeneous solution is $c_1e^{-x} + c_2e^x$, where c_1 and c_2 are constants. As such, the general solution is

$$y = c_1e^{-x} + c_2e^x + \frac{1}{2}(xe^x - \sin x).$$

□

2.3. *Operator Methods*

Definition 2.3 (differential operator). Let x denote independent variable, and y be a dependent variable. Define

$$D_y = \frac{d}{dx}y \quad \text{and} \quad D^n y = \frac{d^n}{dx^n}y = y^{(n)}.$$

Also, let $D^0 y = y$. Given a polynomial

$$L(x) = \sum_{j=0}^{n} a_j x^j \quad \text{where } a_j \text{ are constants,}$$

define a differential operator $L(D)$ by

$$L(D)y = \sum_{j=0}^{n} a_j D^j y.$$

The equation

$$\sum_{j=0}^{n} a_j y^{(j)} = f(x) \quad \text{can be written as} \quad L(D)y = f(x).$$

Proposition 2.2. Let $[L(D)]^{-1} f$ denote any solution of $L(D)y = f(x)$. Then, we have

$$D^{-1}D = DD^{-1} = D^0 \quad \text{and} \quad [L(D)]^{-1}L(D) = L(D)[L(D)]^{-1} = D^0.$$

However, $[L(D)]^{-1} f$ is not unique.

Proof. To see why these properties hold, recall that $D^{-1}f$ means a solution to $y' = f$. Hence, $D^{-1}f$ can be regarded as the integral operator

$$D^{-1}f = \int f.$$

As such, it follows that

$$D^{-1}D = DD^{-1} = \text{identity operator } D^0.$$

As for the second equality, note that a solution to $L(D)y = L(D)f$ is simply f. By definition of $[L(D)]^{-1}$, we have $[L(D)]^{-1}[L(D)f] = f$, so $[L(D)]^{-1}L(D) = D^0$. Lastly, since $[L(D)]^{-1}f$ is a solution to $L(D)y = f(x)$, the result follows. □

More generally, we have the following proposition:

Proposition 2.3. We have the following:

(i)

$$D^{-1}f(x) = \int f(x)\ dx + c$$

(ii)

$$(D-a)^{-1}f(x) = Ce^{ax} + e^{ax-ax}f(x)\ dx$$

(iii)

$$L(D)(e^{ax}f(x)) = e^{ax}L(D+a)f(x)$$

(iv)

$$[L(D)]^{-1}(e^{ax}f(x)) = e^{ax}[L(D+a)]^{-1}f(x)$$

Let $L(x) = (x-r_1)\ldots(x-r_n)$. The solution to $L(D)y = f(x)$ is

$$y = [L(D)]^{-1}f(x) = [(D-r_1)]^{-1}\ldots[(D-r_n)]^{-1}f(x).$$

Then, we obtain the solution by successive integration. Moreover, if the r_j's are distinct, we can write

$$\frac{1}{L(x)} = \frac{A_1}{x-r_1} + \ldots + \frac{A_n}{x-r_n},$$

where the A_j's can be found by the method of partial fractions. As such, the solution is

$$y = \left[[A_1(D-r_1)]^{-1} + \ldots + [A_n(D-r_n)]^{-1}\right]f(x).$$

For the case of repeated roots, let the multiple root be m and the equation to be solved be

$$(D-m)^n y = f(x).$$

To solve this equation, assume a solution of the form $y = e^{mx}v(x)$, where $v(x)$ is to be determined. One can verify that $(D-m)^n e^{mx}v = e^{mx}D^n v$, so $(D-m)^n y = f(x)$ reduces to $D^n v = e^{-mx}f(x)$. Integrating this n times, we obtain

$$v = \int\ldots\int e^{-mx}f(x)\ dx\ldots dx + c_0 + c_1x + \ldots + c_1x^{n-1}$$

and we conclude that

$$(D-m)^{-n}f(x) = e^{mx}\int\ldots\int e^{-mx}f(x)\ dx\ldots dx + c_0 + c_1x + \ldots + c_{n-1}x^{n-1}$$

Example 2.13. Solve

$$\left(D^2-3D+2\right)y=xe^x.$$

Solution. By partial fraction decomposition, we have

$$\frac{1}{D^2-3D+2}=\frac{1}{D-2}-\frac{1}{D-1}.$$

Hence,

$$\begin{aligned}
y&=\left(D^2-3D+2\right)^{-1}(xe^x)\\
&=(D-2)^{-1}(xe^x)-(D-1)^{-1}(xe^x)\\
&=e^{2x}D^{-1}\left(e^{-2x}xe^x\right)-e^xD^{-1}\left(e^{-x}xe^x\right)\\
&=e^{2x}D^{-1}\left(xe^{-x}\right)-e^xD^{-1}(x)\\
&=e^{2x}\left(-xe^{-x}-e^{-x}+c_1\right)-e^x\left(\frac{1}{2}x^2+c_2\right)\\
&=-e^x\left(\frac{1}{2}x^2+x+1\right)+c_1e^{2x}+c_2e^x
\end{aligned}$$

□

Example 2.14. Solve

$$\left(D^3-3D^2+3D-1\right)y=e^x.$$

Solution. By the binomial theorem, we see that the differential equation is equivalent to

$$(D-1)^3y=e^x.$$

As such,

$$\begin{aligned}
y=(D-1)^{-3}e^x&=e^x\left(\iiint e^{-x}e^x\,dx+c_0+c_1x+c_2x^2\right)\\
&=e^x\left(\frac{1}{6}x^3+c_0+c_1x+c_2x^2\right)
\end{aligned}$$

□

Note that if $f(x)$ is a polynomial in x, then

$$(1-D)\left(1+D+D^2+\dots\right)f=f.$$

As such, $(1-D)^{-1}(f)=\left(1+D+D^2+\dots\right)f$. So, if f is a polynomial, we may formally expand $(D-r)^{-1}$ into power series in D and apply it to f. If $\deg f=n$, then it is only necessary to expand $(D-r)^{-1}$ up to D^n.

Example 2.15. Solve

$$\left(D^4-2D^3+D^2\right)y=x^3.$$

Solution. We have

$$\begin{aligned}
y&=\left(D^4-2D^3+D^2\right)^{-1}f\\
&=\frac{1}{D^2(1-D)^2}x^3\\
&=D^{-2}\left(1+2D+3D^2+4D^3+5D^4+6D^5+\dots\right)x^3\\
&=D^{-2}\left(x^3+6x^2+18x+24\right)\\
&=D^{-1}\left(\frac{1}{4}x^4+2x^3+9x^2+24x\right)\\
&=\frac{1}{20}x^5+\frac{1}{2}x^4+3x^3+12x^2
\end{aligned}$$

As such, the general solution is

$$y = (c_1 + c_2 x)\, e^x + (c_3 + c_4 x) + \frac{1}{20}x^5 + \frac{1}{2}x^4 + 3x^3 + 12x^2.$$

□

3. Second-Order Linear Differential Equations

3.1. *Exact Second-Order Equations*

Definition 3.1 (exact equation)**.** The general second-order linear differential equation is of the form

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = f(x).$$

The equation can be written as

$$(p_0y' - p_0'y)' + (p_1y)' + (p_0'' - p_1' + p_2)y = f(x).$$

We say that

$$\text{the differential equation is exact} \quad \text{if} \quad p_0'' - p_1' + p_2 = 0.$$

Theorem 3.1. In relation to Definition 3.1, if the differential equation is exact, we have

$$p_0(x)y' - p_0'(x)y + p_1(x)y = \int f(x)\,dx + c_1.$$

Example 3.1. Find the general solution to the differential equation

$$\frac{1}{x}\frac{d^2y}{dx^2} + \left(\frac{1}{x} - \frac{2}{x^2}\right)\frac{dy}{dx} - \left(\frac{1}{x^2} - \frac{2}{x^3}\right)y = e^x.$$

Solution. Here,

$$p_0(x) = \frac{1}{x} \quad p_1(x) = \frac{1}{x} - \frac{2}{x^2} \quad p_2(x) = \frac{2}{x^3} - \frac{1}{x^2}.$$

One checks that $p_0'' - p_1' + p_2 = 0$. By Theorem 3.1, we have

$$\begin{aligned}\frac{1}{x}\frac{dy}{dx} + \frac{1}{x^2}y + \left(\frac{1}{x} - \frac{2}{x^2}\right)y &= e^x + c_1\\ \frac{dy}{dx} + \left(1 - \frac{1}{x}\right)y &= xe^x + c_1x\end{aligned}$$

By the method of integrating factor, one can deduce that

$$y = \frac{1}{2}xe^x + c_1x + c_2xe^{-x}.$$

□

3.2. *The Adjoint Differential Equation and Integrating Factor*

Recall the differential equation

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = f(x).$$

Suppose this is multiplied by a function $v(x)$ so that the resulting equation is exact. Then, $v(x)$ is said to be an integrating factor of the differential equation. That is,

$$(p_0v)'' - (p_1v)' + p_2v = 0.$$

This is a differential equation in terms of v, which is, more explicitly, by the product rule,

$$p_0(x)v'' + (2p_0'(x) - p_1(x))v' + (p_0''(x) - p_1'(x) + p_2(x))v = 0.$$

This equation is called the adjoint of the original differential equation. Thus, we see that a function $v(x)$ is an integrating factor for a given differential equation if and only if it is a solution of the adjoint equation. Note that the adjoint is in turn found to be the associated homogeneous equation of the original differential equation, thus each is the adjoint of the other.

In this case, a first integral is

$$v(x)p_0(x)y' - (v(x)p_0(x))'y + v(x)p_1(x)y = \int v(x)f(x)\,dx + c.$$

Example 3.2. Find the general solution of the differential equation

$$(x^2 - x)y'' + (2x^2 + 4x - 3)y' + 8xy = 1.$$

Solution. The adjoint of this equation is

$$(x^2 - x)v'' - (2x^2 - 1)v' + (4x - 2)v = 0.$$

By trying powers of x^m, one checks that $v = x^2$ satisfies this equation. In turn, x^2 is an integrating factor of the original differential equation. Multiplying it by x^2, we obtain

$$(x^4 - x^3)y'' + (2x^4 + 4x^3 - 3x^2)y' + 8x^3y = x^2.$$

Hence, a first integral is

$$(x^4 - x^3)y' - (4x^3 - 3x^2)y + (2x^4 + 4x^3 - 3x^2)y = \int x^2\,dx + c.$$

Upon simplification, we have

$$y' + \frac{2x}{x-1}y = \frac{1}{3(x-1)} + \frac{c}{x^3(x-1)}.$$

One can use the method of integrating factor to deduce that

$$y = \frac{1}{(x-1)^2}\left(\frac{x}{6} - \frac{1}{4} + \frac{c_1}{x^2} + c_2e^{-2x}\right).$$

□

Example 3.3. The differential equation

$$(p(x)y')' + q(x)y + \lambda r(x)y = 0$$

is called a Sturm-Liouville equation, where λ is a real parameter (often called an eigenvalue parameter), and the functions $p(x), q(x), r(x)$ are given with $p(x) > 0$ and $r(x) > 0$ on some interval. It is a well-known fact that the Sturm-Liouville operator is *self-adjoint* under suitable boundary conditions. This means that the adjoint of the differential operator

$$L[y] := -(p(x)y')' + q(x)y$$

is the operator itself, in the sense of the inner product

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}r(x)\,dx.$$

To see why this is the case, note that we can write the equation as

$$-(p(x)y')' + q(x)y = \lambda r(x)y.$$

The formal adjoint of the operator L can be computed using integration by parts and turns out to be identical to L when the functions and boundary conditions are chosen appropriately. Hence, Sturm-Liouville operators are self-adjoint in the formal sense.

3.3. *Lagrange's Identity and Green's Formula*

Recall the differential equation $p_0(x)y'' + p_1(x)y' + p_2(x)y = f(x)$. Let L be the differential operator given by the left side of this equation, i.e. $L[y] = p_0(x)y'' + p_1(x)y' + p_2(x)y$. The formal adjoint of L is the differential operator defined by $L^+[y] = (p_0(x)y)'' - (p_1(x)y')' + (p_2(x)y$, where p_0'', p_1' and p_2 are continuous on an interval $[a,b]$. Let u and v be functions having continuous second-order derivatives on $[a,b]$. Direct simplification yields Lagrange's identity (Theorem 3.2) relating L and L^+.

Theorem 3.2 (Lagrange's identity). Let $P(u,v) = up_1v - u(p_0v)' + u'p_0v$. Then,

$$L[u]v - uL^+[v] = \frac{d}{dx}[P(u,v)].$$

Integrating both sides of Lagrange's identity yields Green's formula (Corollary 3.1).

Corollary 3.1 (Green's formula). We have

$$\int_a^b (L[u]v - uL^+[v])\,dx = P(u,v)(b) - P(u,v)(a).$$

3.4. *Regular Boundary Value Problem*

The problem of finding a solution to a second-order linear differential equation

$$y'' + p(x)y' + q(x)y = f(x) \quad \text{where } x \in (a,b)$$

satisfying the boundary conditions

$$a_{11}y(a) + a_{12}y'(a) + b_{11}y(b) + b_{12}y'(b) = d_1$$
$$a_{21}y(a) + a_{22}y'(a) + b_{21}y(b) + b_{22}y'(b) = d_2$$

is known as a two-point boundary value problem. When $d_1 = d_2 = 0$, the boundary conditions are said to be homogeneous; otherwise we refer to them as non-homogeneous.

Proposition 3.1. For the homogeneous equation $y'' + p(x)y' + q(x)y = 0$ for all $a < x < b$ with homogeneous boundary condition, we have the following properties:

- **(i)** If $\phi(x)$ is a non-trivial solution, then so is $c\phi(x)$ for any constant c. That is, the differential equation has a one-parameter family of solutions.
- **(ii)** If the differential equation has two linearly independent solutions $\phi_1(x)$ and $\phi_2(x)$, then any linear combination $c_1\phi_1(x) + c_2\phi_2(x)$ is also a solution to the differential equation for any constants c_1, c_2. Thus, the equation has a two-parameter family of solutions.
- **(iii)** The remaining possibility is that $\phi(x) = 0$ is the unique solution to the differential equation

For the case where the equation is non-homogeneous, there is a possibility that it has no solution. We give some examples to illustrate these cases.

Example 3.4. Find all solutions to the boundary value problem

$$y'' + 2y' + 5y = 0 \quad \text{where } y(0) = 2 \text{ and } y\left(\frac{\pi}{4}\right) = e^{-\pi/4}.$$

Solution. The general solution to the equation $y'' + 2y' + 5y = 0$ is $y = c_1e^{-x}\cos 2x + c_2e^{-x}\sin 2x$. Substituting the boundary conditions, one can deduce that $c_1 = 2$ and $c_2 = 1$ so the boundary value problem has the unique solution $y = 2e^{-x}\cos 2x + e^{-x}\sin 2x$. □

Example 3.5. Find all solutions to the boundary value problem

$$y'' + y = \cos 2x \quad \text{where } y'(0) = 0 \text{ and } y'(\pi) = 0.$$

Solution. The general solution to the equation $y'' + y = \cos 2x$ is $y = c_1 \cos x + c_2 \sin x - \frac{1}{3}\cos 2x$. Thus, $y' = -c_1 \sin x + c_2 \cos x + \frac{2}{3}\sin 2x$. Substituting the boundary conditions, we have $c_2 = 0$, so the boundary value problem has a one-parameter family of solutions $y = c_1 \cos x - \frac{1}{3}\cos 2x$, where $c_1 \in \mathbb{R}$. □

Example 3.6. Find all solutions to the boundary value problem

$$y'' + 4y = 0 \quad \text{where } y(-\pi) = y(\pi) \text{ and } y'(-\pi) = y'(\pi).$$

Solution. The general solution to the equation $y'' + 4y = 0$ is $y = c_1 \cos 2x + c_2 \sin 2x$. One checks that the boundary value problem has a two-parameter family of solutions $y = c_1 \cos 2x + c_2 \sin 2x$, where $c_1, c_2 \in \mathbb{R}$. □

Example 3.7. Find all solutions to the boundary value problem

$$y'' + 4y = 4x \quad \text{where } y(-\pi) = y(\pi) \text{ and } y'(-\pi) = y'(\pi).$$

Solution. The general solution to the equation $y'' + 4y = 4x$ is $y = x + c_1 \cos 2x + c_2 \sin 2x$. Since $y(-\pi) = -\pi + c_1$ and $y(\pi) = \pi + c_1$, there are no solutions satisfying $y(-\pi) = y(\pi)$. Hence, the boundary value problem has no solution. □

3.5. *Regular Sturm-Liouville Boundary Value Problem*

Let $L[y] = (p(x)y')' + q(x)y$. Consider the regular Sturm-Liouville boundary value problem $L[y] + \lambda r(x)y = 0$ with $a < x < b$, where $a_1 y(a) + a_2 y'(a) = 0$ and $b_1 y(b) + b_2 y'(b) = 0$. Here, $p(x), p'(x), q(x), r(x)$ are continuous functions on $[a,b]$ and $p(x), r(x) > 0$ on $[a,b]$. We will exclude the cases where $a_1 = a_2 = 0$ or $b_1 = b_2 = 0$.

Let u and v be functions that have continuous second derivatives on $[a,b]$ and satisfy the boundary conditions. The boundary conditions imply that $W(u,v)(b) = W(u,v)(a) = 0$. This is because the system of equations $a_1 u(a) + a_2 u'(a) = 0$, $a_1 v(a) + a_2 v'(a) = 0$ has non-trivial solutions in a_1 and a_2 since a_1 and a_2 are not both zero. Therefore the determinant of the system $W(u,v)(a)$ must be zero. Similarly $W(u,v)(b) = 0$.

Thus by Green's formula (Corollary 3.1), we have $([L]u, v) = (u, [L]v)$. Therefore, L is a self-adjoint operator with domain equal to the set of functions that have continuous second derivatives on $[a,b]$ and satisfy the boundary conditions. Self-adjoint operators are like symmetric matrices in that their eigenvalues are always real.

Here we only require u and v satisfy the boundary conditions in (3.5.13) but not necessarily the differential equation $L[y] + \lambda r(x)y = 0$. However, if u and v satisfy the differential equation, then $L[u]v = uL[v] = -\lambda ruv$ and hence,

$$([L]u, v) = (u, [L]v)$$

too.

The regular Sturm-Liouville boundary value problem involves a parameter λ. The objective is to determine for which values of λ, the equation has non-trivial solutions satisfying the given boundary conditions. Such problems are called *eigenvalue problems*. The non-trivial solutions are called *eigenfunctions*, and the corresponding number λ an *eigenvalue*. If all the eigenfunctions associated with a particular eigenvalue are just scalar multiples of each other, then the eigenvalue is called *simple*.

Theorem 3.3. All the eigenvalues of the regular Sturm-Liouville boundary value problem are real, have real-valued eigenfunctions and simple.

Two real-valued functions f and g defined on $[a,b]$ are said to be *orthogonal* with respect to a positive weight function $r(x)$ on the interval $[a,b]$ if

$$\int_a^b f(x)g(x)r(x)\,dx = 0.$$

Theorem 3.4. Eigenfunctions that correspond to distinct eigenvalues of the regular Sturm-Liouville boundary value problem in are orthogonal with respect to the weight function $r(x)$ on $[a,b]$.

Theorem 3.5. The eigenvalues of the regular Sturm-Liouville boundary value problem form a countable, increasing sequence

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots, \quad \text{with} \quad \lim_{n\to\infty} \lambda_n = +\infty.$$

Example 3.8. Consider the regular Sturm-Liouville boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = y(\pi) = 0.$$

When $\lambda \le 0$, the boundary value problem has only the trivial solution $y = 0$. Thus for $\lambda \le 0$, it is not an eigenvalue. Let us consider $\lambda > 0$. The general solution to the equation $y'' + \lambda y = 0$ is given by

$$y = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

Now $y(0) = 0$ implies that $A = 0$, and $y(\pi) = 0$ implies that $B\sin(\sqrt{\lambda}\pi) = 0$. Since we are looking for nontrivial solutions, we must have $B \ne 0$ so that $\sin(\sqrt{\lambda}\pi) = 0$. Therefore $\lambda = n^2$, where $n = 1,2,3,\ldots$ with corresponding eigenfunctions $\phi_n(x) = B_n \sin(nx)$. One can easily verify that $(\phi_m, \phi_n) = 0$ for $m \ne n$.

Thus, associated to a regular Sturm-Liouville boundary value problem, there is a sequence of orthogonal eigenfunctions $\{\phi_n\}$ defined on $[a,b]$. We can use these eigenfunctions to form an *orthonormal* system with respect to $r(x)$ simply by normalising each eigenfunction ϕ_n so that

$$\int_a^b \phi_n^2(x)r(x)\,dx = 1.$$

Now suppose $\{\phi_n\}$ is orthonormal with respect to a positive weight function $r(x)$ on $[a,b]$, that is

$$\int_a^b \phi_n(x)\phi_m(x)r(x)\,dx = \begin{cases} 0, & n \ne m; \\ 1, & n = m. \end{cases}$$

Then with any piecewise continuous function f on $[a,b]$, we can identify an orthogonal expansion

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x),$$

where

$$c_n = \int_a^b f(x)\phi_n(x)r(x)\,dx.$$

For instance, the eigenfunctions $\phi_n(x) = \sin nx$ gives rise to the Fourier sine series expansion. Like the theory of Fourier series, the eigenfunction expansion of f converges uniformly to f on $[a,b]$ under suitable conditions.

Theorem 3.6. Let $\{\phi_n\}$ be an orthonormal system of eigenfunctions for the regular Sturm-Liouville boundary value problem. Let f be a continuous function on $[a,b]$ such that f' is piecewise continuous on $[a,b]$, and f satisfies the boundary conditions. Then

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \quad a \leq x \leq b$$

where

$$c_n = \int_a^b f(x)\phi_n(x)r(x)\,dx.$$

Furthermore, the eigenfunction expansion converges uniformly on $[a,b]$.

3.6. *Non-Homogeneous Boundary Value Problems*

Of interest now are non-homogeneous regular Sturm-Liouville boundary value problems with homogeneous boundary conditions. Let $L[y] = f(x)$ where $a < x < b$ and $a_1 y(a) + a_2 y'(a) = 0$ and $b_1 y(b) + b_2 y'(b) = 0$, where $L[y] = (p(x)y')' + q(x)y, p(x), p'(x), q(x)$ are continuous functions on $[a,b]$ and $p(x) > 0$ on $[a,b]$, and $f(x)$ is continuous on $[a,b]$.

Theorem 3.7. The non-homogeneous problem has a unique solution if and only if the homogeneous problem has only the trivial solution.

Example 3.9. Find the Green's function $G(x,t)$ for the boundary value problem

$$y'' = f \quad \text{where } y(0) = 0 \text{ and } y(\pi) = 0.$$

Solution. A general solution to $y'' = 0$ is $y = Ax + B$. Thus, $y_1 = x$ is a solution satisfying $y(0) = 0$, and $y = \pi - x$ is a solution satisfying $y(\pi) = 0$. Also, y_1 and y_2 are linearly independent. As such, the Green's function is

$$G(x,t) = \begin{cases} \frac{t(x-\pi)}{\pi} & \text{if } 0 \leq t \leq x; \\ \frac{x(t-\pi)}{\pi} & \text{if } x \leq t \leq \pi. \end{cases}$$

Hence, for $f(x) = -6x$, the solution is given by

$$y(x) = \int_0^{\pi} G(x,t)f(t)\,dt = x\left(\pi^2 - x^2\right).$$

Since the homogeneous problem $y'' = 0$, where $y(0) = 0$ and $y(\pi) = 0$, has only the trivial solution, the above solution to the homogeneous problem $y'' = -6x$ with $y(0) = 0$ and $y(\pi) = 0$ is unique. □

We can strengthen Theorem 3.7 as follows (Theorem 3.8):

Theorem 3.8 (Fredholm). The non-homogeneous problem has a solution if and only if for every solution $y(x)$ of the homogeneous problem, we have

$$\int_a^b f(t)y(t)\,dt = 0.$$

Example 3.10. Show that the boundary value problem

$$y'' + y' + \frac{5}{2}y = f \quad \text{where } y(0) = 0 \text{ and } y\left(\frac{2\pi}{3}\right) = 0$$

has a solution if

$$\int_0^{2\pi/3} e^{x/2} \sin\left(\frac{3x}{2}\right) f(x)\,dx = 0.$$

Solution. Consider the homogeneous boundary value problem $y'' + y' + \frac{5}{2}y = 0$, with $y(0) = 0$ and $y\left(\frac{2\pi}{3}\right) = 0$. The general solution of the homogeneous equation is $y = e^{-x/2}\left(A\cos\left(\frac{3x}{2}\right) + B\sin\left(\frac{3x}{2}\right)\right)$. Since $y(0) = 0$, then $A = 0$. Thus, $y = Be^{-x/2}\sin\left(\frac{3x}{2}\right)$. This also satisfies $y\left(\frac{2\pi}{3}\right) = 0$. Thus, the homogeneous boundary value problem has a one parameter family of solutions. Note that the given differential equation is not self-adjoint, but we can convert it into a self-adjoint equation by multiplying throughout by a factor e^x. So, we may write the problem as

$$(e^x y')' + \frac{5}{2}e^x y = e^x f(x) \quad \text{where } y(0) = 0 \text{ and } y\left(\frac{2\pi}{3}\right) = 0.$$

By Fredholm's theorem (Theorem 3.8), this problem has a solution if and only if

$$\int_0^{2\pi/3} Be^{-x/2}\sin\left(\frac{3x}{2}\right) \cdot e^x f(x)\, dx = 0$$

which is equivalent to the condition mentioned in the question. □

3.7. *The Sturm Separation Theorem and the Sturm Comparison Theorem*

Consider the homogeneous second-order linear differential equation $y'' + P(x)y' + Q(x)y = 0$. It is rarely possible to solve this equation in general. However, by studying the properties of the coefficient functions, it is sometimes possible to describe the behaviour of the solutions. One of the important characteristics that is of interest is the number of zeros of a solution to the mentioned differential equation. If a function has an infinite number of zeros in an interval $[a, \infty)$, we say that the function is oscillatory. Hence, studying the oscillatory behaviour of a function means investigating the number and locations of its zeros.

As a motivation, consider the familiar equation $y'' + y = 0$, which has two linearly independent solutions $s(x) = \sin x$ satisfying $y(0) = 0$ and $y'(0) = 1$, and $c(x) = \cos x$ satisfying $y(0) = 1$ and $y'(0) = 0$ respectively. The positive zeros of $s(x)$ and $c(x)$ are $\pi, 2\pi, 3\pi, \ldots$ and $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ respectively. Note that the zeros of $s(x)$ and $c(x)$ are said to interlace one another in the sense that between two successive zeros of $s(x)$, there is a zero of $c(x)$ and vice versa.

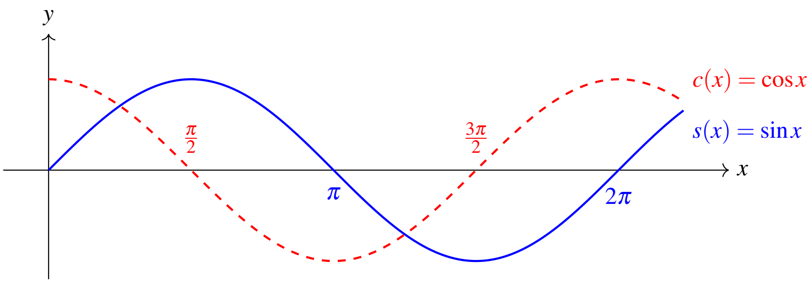

Figure 1: Interlacing of zeros of two linearly independent solutions

This property is described by the Sturm separation theorem (Theorem 3.9).

Theorem 3.9 (Sturm separation theorem). If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions to

$$y'' + P(x)y' + Q(x)y = 0,$$

then the zeros of these functions are distinct and occur alternatively in the sense that $y_1(x)$ vanishes exactly once between any two successive zeros of $y_2(x)$ and conversely.

Corollary 3.2. Suppose one non-trivial solution to $y''+P(x)y'+Q(x)y=0$ is oscillatory on $[a,\infty)$. Then, all solutions are oscillatory.

Now, note that the equation $y''+P(x)y'+Q(x)y=0$ can be written in the form $u''+q(x)u=0$ by setting $y=uv$, where $v=e^{-\frac{1}{2}\int P\,dx}$ and $q(x)=Q(x)-\frac{1}{4}(P(x))^2-\frac{1}{2}P'(x)$. We refer to the original differential equation as the standard form and the transformed one as the normal form. Since $v(x)>0$ for all x, the above transformation has no effect on the zeros of the solutions, and therefore leaves unaltered the oscillation behaviour of the solutions.

Example 3.11. By using the substitution $y=e^{-\frac{x^3}{3}}u$, find the general solution of the equation

$$y''+2x^2y'+(x^4+2x+1)y=0.$$

Show that the distance between two consecutive zeros of any non-trivial solution is π.

Solution. We leave it to the reader to deduce that

$$y=Ae^{-\frac{x^3}{3}}\sin(x-\theta).$$

Setting $y=0$, we have $\sin(x-\theta)=0$, so the result follows. □

Theorem 3.10 (Sturm comparison theorem). Let y_1 be a non-trivial solution to $y''+q_1(x)y=0$ and y_2 be a non-trivial solution to $y''+q_2(x)y=0$, where $a<x<b$. Assume that $q_2(x)\geq q_1(x)$ for all $a<x<b$. If x_1 and x_2 are two consecutive zeros of y_1 on (a,b) with $x_1<x_2$, then there exists a zero of y_2 in (x_1,x_2), unless $q_2(x)=q_1(x)$ on $[x_1,x_2]$ in which case y_1 and y_2 are linearly dependent.

Corollary 3.3. Suppose $q(x)\leq 0$ for all $x\in[a,b]$. If y is a non-trivial solution to $y''+q(x)y=0$ on $[a,b]$, then y has at most one zero on $[a,b]$.

Example 3.12. The equation

$$x^2y''+xy'+(x^2-p^2)y=0 \quad \text{where } x>0$$

is called Bessel's equation of order p. For $a>0$, we shall discuss the number of zeros in the interval $[a,a+\pi)$. The substitution $y=ux^{-\frac{1}{2}}$ transforms the mentioned equation to the one as follows:

$$\frac{d^2u}{dx^2}+\left(1-\frac{p^2-\frac{1}{4}}{x^2}\right)u=0 \quad \text{where } x>0.$$

Since $y=ux^{-\frac{1}{2}}$, the distribution of zeros of a solution y to the old differential equation is the same as the corresponding solution u to the new equation. We shall compare the solutions to the new equation with those to $u''+u=0$. Observe that $u(x)=A\sin(x-a)$ is a solution to $u''+u=0$ and has zeros at a and $a+\pi$.

- **Case 1:** Suppose $p>1/2$. Then, $4p^2-1>0$ so $1-\frac{p^2-\frac{1}{4}}{x^2}<1$ for all $x\in[a,a+\pi)$. By the Sturm comparison theorem (Theorem 3.10), a solution to the new equation cannot have more than one zero in $[a,a+\pi)$ because $u(x)=A\sin(x-a)$ does not have a zero in $(a,a+\pi)$.
- **Case 2:** Next, suppose $0\leq p<1/2$. Then, $4p^2-1<0$ so that $1-\frac{p^2-\frac{1}{4}}{x^2}>1$ for $x\in[a,a+\pi)$. Again by the Sturm comparison theorem (Theorem 3.10), a solution to the new equation must have a zero in $(a,a+\pi)$ since a and $a+\pi$ are consecutive zeros of $u(x)=A\sin(x-a)$.
- **Case 3:** If $p=1/2$, then the differential equation reduces to $u''+u=0$, which has the general solution $u(x)=A\sin(x-a)$. Consequently, it has exactly one zero in $[a,a+\pi)$.

4. Linear Differential Systems

4.1. *Linear Systems*

The following system is called a linear differential system of first order in normal form:

$$\begin{cases} x_1' &= a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + g_1(t) \\ \vdots &= \vdots \\ x_n' &= a_{n1}(t)x_1 + \cdots + a_{nn}(t)x_n + g_n(t) \end{cases}$$

where $a_{ij}(t)$ and $g_j(t)$ are continuous functions of t and $'$ denotes differentiation with respect to t. Denote

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix} \quad \mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix}.$$

Define

$$\mathbf{x}' = \begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} \quad \int \mathbf{x}(t)\,dt = \begin{pmatrix} \int x_1(t)\,dt \\ \vdots \\ \int x_n(t)\,dt \end{pmatrix}.$$

Then, the linear system can be written in the matrix form:

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{g}(t).$$

When $\mathbf{g} \equiv 0$, the above equation is reduced to $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$. This equation is called a homogeneous differential system.

Example 4.1. The system

$$\begin{aligned} x_1' &= 2x_1 + 3x_2 + 3t \\ x_2' &= -x_1 + x_2 - 7\sin t \end{aligned}$$

is equivalent to

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 3t \\ -7\sin t \end{pmatrix}.$$

Example 4.2. Given a second order system

$$\begin{aligned} \frac{d^2x}{dt^2} &= x + 2y + 3t \\ \frac{d^2y}{dt^2} &= 4x + 5y + 6t \end{aligned}$$

let $u = x'$ and $v = y'$. Then we have

$$\begin{cases} x' = u, \\ u' = x + 2y + 3t, \\ y' = v, \\ v' = 4x + 5y + 6t. \end{cases}$$

Next, we consider the initial value problem:

$$\begin{aligned} \mathbf{x}' &= \mathbf{A}(t)\mathbf{x} + \mathbf{g}(t) \\ \mathbf{x}(t_0) &= \mathbf{x}_0, \end{aligned}$$

where $\mathbf{x}_0$ is a constant vector.

Theorem 4.1. Assume that $\mathbf{A}(t)$ and $\mathbf{g}(t)$ are continuous on an open interval $a < t < b$ containing t_0. Then, for any constant vector $\mathbf{x}_0$, the initial value problem

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{g}(t)$$
$$\mathbf{x}(t_0) = \mathbf{x}_0,$$

has a solution $\mathbf{x}(t)$ defined on this interval. This solution is unique. Also, if $\mathbf{A}(t)$ and $\mathbf{g}(t)$ are continuous on $\mathbb{R}$, then for any $t_0 \in \mathbb{R}$ and $\mathbf{x}_0 \in \mathbb{R}^n$, the initial value problem has a unique solution $\mathbf{x}(t)$ defined on $\mathbb{R}$.

4.2. *Homogeneous Linear Systems*

Now, we assume $\mathbf{A} = (a_{ij}(t))$ is a continuous $n \times n$ matrix-valued function on (a,b).

Lemma 4.1. Let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ be two solutions of $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ on (a,b). Then for any constants c_1, c_2, the function $\mathbf{z}(t) = c_1\mathbf{x}(t) + c_2\mathbf{y}(t)$ is also a solution of the differential system on (a,b).

Definition 4.1. $\mathbf{x}_1(t), \dots, \mathbf{x}_r(t)$ are linearly dependent on (a,b) if there exist constants $c_1, \dots, c_r$, not all zero, such that

$$c_1\mathbf{x}_1(t) + \cdots + c_r\mathbf{x}_r(t) = 0 \quad \text{for all } t \in (a,b).$$

Otherwise, they are linearly independent.

Lemma 4.2. A set of solutions $\mathbf{x}_1(t), \dots, \mathbf{x}_r(t)$ of $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ are linearly dependent on (a,b) *if and only if* $\mathbf{x}_1(t_0), \dots, \mathbf{x}_r(t_0)$ are linearly dependent vectors for any fixed $t_0 \in (a,b)$.

Theorem 4.2. The following hold:

(i) The differential system in $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ has n linearly independent solutions

(ii) Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be any set of n linearly independent solutions of $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ on (a,b). Then the general solution of the differential system is given by

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$$

where c_j are arbitrary constants.

Definition 4.2 (Wronskian). The Wronskian of n vector-valued functions

$$\mathbf{x}_1(t) = \begin{pmatrix} x_{11}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix} \quad \dots \quad \mathbf{x}_n(t) = \begin{pmatrix} x_{1n}(t) \\ \vdots \\ x_{nn}(t) \end{pmatrix}$$

is defined as the determinant

$$W(t) = W(\mathbf{x}_1, \dots, \mathbf{x}_n)(t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{vmatrix}.$$

Theorem 4.3. The following hold:

(i) The Wronskian of n solutions of $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ is either identically zero or nowhere zero in (a,b)

(ii) n solutions of $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ are linearly dependent in (a,b) if and only if their Wronskian is identically zero in (a,b)

A set of n linearly independent solutions of $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ is called a fundamental set of solutions, or a basis of solutions. Let

$$\mathbf{x}_1(t) = \begin{pmatrix} x_{11}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix} \quad \cdots \quad \mathbf{x}_n(t) = \begin{pmatrix} x_{1n}(t) \\ \vdots \\ x_{nn}(t) \end{pmatrix}$$

be a fundamental set of solutions of $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ on (a,b). The matrix-valued function

$$\Phi(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{pmatrix}$$

is called a fundamental matrix of the linear system on (a,b).

Theorem 4.4. Let $\Phi(t)$ be a fundamental matrix of $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ on (a,b). Then the general solution of the linear system is given by

$$\mathbf{x}(t) = \Phi(t)\mathbf{c},$$

where $\mathbf{c} = (c_1, \ldots, c_n)$ is an arbitrary constant vector.

4.3. *Non-Homogeneous Linear Systems*

Now, we consider the solutions of the non-homogeneous system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{g}(t)$, where $\mathbf{A} = (a_{ij}(t))$ is a continuous $n \times n$ matrix-valued function and $\mathbf{g}(t)$ is a continuous vector-valued function, both defined on the interval (a,b).

Theorem 4.5. Let $\mathbf{x}_p(t)$ be a particular solution of $\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{g}(t)$, and $\Phi(t)$ be a fundamental matrix of the associated homogeneous system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$. Then the general solution of $\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{g}(t)$ is given by

$$\mathbf{x}(t) = \Phi(t)\mathbf{c} + \mathbf{x}_p(t)$$

where $\mathbf{c}$ is an arbitrary constant vector.

4.4. *Homogeneous Linear Systems with Constant Coefficients*

Consider a homogeneous linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = (a_{ij})$ is a constant $n \times n$ matrix. We shall try to find a solution of the linear system of the form $\mathbf{x}(t) = e^{\lambda t}\mathbf{k}$, where $\mathbf{k}$ is a constant vector, where $\mathbf{k} \neq 0$. Plugging it into the system, we see that $\mathbf{A}\mathbf{k} = \lambda\mathbf{k}$.

Definition 4.3. Assume that a number λ and a vector $\mathbf{k} \neq 0$ satisfy $\mathbf{A}\mathbf{k} = \lambda\mathbf{k}$, then we call λ an eigenvalue of $\mathbf{A}$, and $\mathbf{k}$ an eigenvector associated with λ.

Recall the following. Let $\mathbf{A}$ be an $n \times n$ matrix and $\lambda_1, \ldots, \lambda_k$ be the distinct roots of the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. Then there exist positive integers $m_1, \ldots, m_k$, such that

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (-1)^n(\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k},$$

and

$$m_1 + m_2 + \cdots + m_k = n.$$

m_j is called the algebraic multiplicity (or simply multiplicity) of the eigenvalue λ_j. The number of linearly independent eigenvectors of $\mathbf{A}$ associated with λ_j is called the geometric multiplicity, denoted $\mu(\lambda_j)$. We always have

$$\mu(\lambda_j) \le m_j.$$

If $\mu(\lambda_j) = m_j$, we say λ_j is quasi-simple. If $m_j = 1$, we say λ_j is a simple eigenvalue.

Theorem 4.6. If λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{k}$ is an associated eigenvector, then

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{k}$$

is a solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Let $\mathbf{A}$ be a real matrix. If λ is a complex eigenvalue of $\mathbf{A}$, and $\mathbf{k}$ is an eigenvector associated with λ, then

$$\mathbf{x}_1 = \mathrm{Re}(e^{\lambda t}\mathbf{k}) \quad \mathbf{x}_2 = \mathrm{Im}(e^{\lambda t}\mathbf{k})$$

are two linearly independent real solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Theorem 4.7. If $\mathbf{A}$ has n linearly independent eigenvectors $\mathbf{k}_1, \ldots, \mathbf{k}_n$ associated with eigenvalues $\lambda_1, \ldots, \lambda_n$, respectively, then

$$\Phi(t) = \left(e^{\lambda_1 t}\mathbf{k}_1, \ldots, e^{\lambda_n t}\mathbf{k}_n\right)$$

is a fundamental matrix of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and the general solution is given by

$$\mathbf{x}(t) = \Phi(t)\mathbf{c} = c_1 e^{\lambda_1 t}\mathbf{k}_1 + \cdots + c_n e^{\lambda_n t}\mathbf{k}_n$$

where $\mathbf{c} = (c_1, \ldots, c_n)$ is an arbitrary constant vector.

Example 4.3. Solve the system

$$\mathbf{x}' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\mathbf{x}.$$

Solution. The coefficient matrix has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -4$ with respective corresponding eigenvectors $(1,1)$ and $(1,-1)$. So, the general solution is

$$\mathbf{x} = c_1 e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

□

Example 4.4. Solve the system

$$\mathbf{x}' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\mathbf{x} + e^{-2t}\begin{pmatrix} -6 \\ 2 \end{pmatrix}.$$

Solution. We first solve the associated homogeneous system which yields two linearly independent solutions $\mathbf{x}_1(t) = (e^{-2t}, e^{-2t})$ and $\mathbf{x}_2(t) = (e^{-4t}, -e^{-4t})$. The fundamental matrix is

$$\Phi(t) = (\mathbf{x}_1(t), \mathbf{x}_2(t)) = \begin{pmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{pmatrix}$$

We then use the method of variation of parameters. Let $\mathbf{g}(t) = e^{-2t}(-6,2)$. Then,

$$\mathbf{u}(t) = \int_0^t \Phi^{-1}(s)\mathbf{g}(s)\, ds = \begin{pmatrix} -2t \\ -2e^{2t}+2 \end{pmatrix}$$

so

$$\Phi(t)\mathbf{u}(t) = 2e^{-2t}\begin{pmatrix} -t-1 \\ -t+1 \end{pmatrix} + 2e^{-4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Combining the homogeneous solution with the particular solution yields the general solution. □

Example 4.5. Solve

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\mathbf{x}.$$

Solution. The coefficient matrix has eigenvalues $\pm 2i$. For $\lambda = 2i$, we have an eigenvector $\mathbf{k} = (1,2i)$. As such, consider

$$e^{2it}\begin{pmatrix} 1 \\ 2i \end{pmatrix} = \begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix} + i\begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix}.$$

Hence, the general solution is

$$\mathbf{x}(t) = c_1\begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix} + c_2\begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix}.$$

□

Example 4.6. Solve

$$\begin{aligned} x' &= -3x+4y-2z \\ y' &= x+z \\ z' &= 6x-6y+5z \end{aligned}$$

Consider the matrix

$$\begin{pmatrix} -3 & 4 & -2 \\ 1 & 0 & 1 \\ 6 & -6 & 5 \end{pmatrix}$$

which has eigenvalues $2,1,-1$ and the respective corresponding eigenvectors are $(0,1,2),(1,1,0),(1,0,-1)$. Hence, the general solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1e^{2t}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + c_2e^{t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3e^{-t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Lemma 4.3. Assume λ is an eigenvalue of $\mathbf{A}$ with algebraic multiplicity $m > 1$. Then the system $(\mathbf{A}-\lambda\mathbf{I})^m\mathbf{v} = \mathbf{0}$ has exactly m linearly independent solutions.

Theorem 4.8. Assume that λ is an eigenvalue of $\mathbf{A}$ with algebraic multiplicity $m > 1$. Let $\mathbf{v}_0 \neq 0$ be a solution of $(\mathbf{A}-\lambda\mathbf{I})^m\mathbf{v} = \mathbf{0}$. Define

$$\mathbf{v}_l = (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_{l-1} \quad l = 1,2,\cdots,m-1,$$

and let

$$\mathbf{x}(t) = e^{\lambda t}\left[\mathbf{v}_0 + t\mathbf{v}_1 + \frac{t^2}{2}\mathbf{v}_2 + \cdots + \frac{t^{m-1}}{(m-1)!}\mathbf{v}_{m-1}\right].$$

Then $\mathbf{x}(t)$ is a solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Let $\mathbf{v}_0^{(1)}, \cdots, \mathbf{v}_0^{(m)}$ be m linearly independent solutions of $(\mathbf{A}-\lambda\mathbf{I})^m\mathbf{v} = \mathbf{0}$. They generate m linearly independent solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Note that for $\mathbf{v}_l = (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_{l-1}$, where $1 \le l \le m-1$, we always have

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_{m-1} = \mathbf{0}.$$

If $\mathbf{v}_{m-1} \neq \mathbf{0}$, then $\mathbf{v}_{m-1}$ is an eigenvector of $\mathbf{A}$ associated with the eigenvalue λ. In practice, to find the solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ associated with an eigenvalue λ of multiplicity m, we first solve $(\mathbf{A}-\lambda\mathbf{I})^m\mathbf{v} = \mathbf{0}$ and find m linearly independent solutions $\mathbf{v}_0^{(1)}, \mathbf{v}_0^{(2)}, \ldots, \mathbf{v}_0^{(m)}$. For each of these vectors, say $\mathbf{v}_0^{(k)}$, we compute the iteration sequence

$$\mathbf{v}_l^{(k)} = (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_{l-1}^{(k)} \quad \text{where } l = 1, 2, \ldots.$$

There is an integer $0 \le j \le m-1$ (j depends on the choice of $\mathbf{v}_0^{(k)}$) such that

$$\mathbf{v}_j^{(k)} \neq \mathbf{0} \quad \text{and} \quad (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_j^{(k)} = \mathbf{0}.$$

Thus $\mathbf{v}_j$ is an eigenvector of $\mathbf{A}$ associated with the eigenvalue λ. Then the iteration stops and yields a solution

$$\mathbf{x}^{(k)}(t) = e^{\lambda t}\left[\mathbf{v}_0^{(k)} + t\mathbf{v}_1^{(k)} + \frac{t^2}{2}\mathbf{v}_2^{(k)} + \cdots + \frac{t^j}{j!}\mathbf{v}_j^{(k)}\right].$$

Example 4.7. Solve $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & -4 \end{pmatrix}.$$

Solution. The eigenvalues of $\mathbf{A}$ are $\lambda_1 = -3$ with with multiplicity 2, and $\lambda_2 = 0$ which is a simple eigenvalue. For the double eigenvalue $\lambda_1 = -3$, we solve

$$(\mathbf{A}+3\mathbf{I})^2\mathbf{v} = \begin{pmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 1 & 1 & 1 \end{pmatrix}\mathbf{v} = \mathbf{0}$$

and find two linearly independent solutions $\mathbf{v}_0^{(1)} = (1, 0, -1)$ and $\mathbf{v}_0^{(2)} = (0, 1, -1)$. So,

$$\mathbf{v}_1^{(1)} = (\mathbf{A}+3\mathbf{I})\mathbf{v}_0^{(1)} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} \quad \mathbf{x}^{(1)} = e^{-3t}\left(\mathbf{v}_0^{(1)} + t\mathbf{v}_1^{(1)}\right) = e^{-3t}\left[\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}\right]$$

$$\mathbf{v}_1^{(2)} = (\mathbf{A}+3\mathbf{I})\mathbf{v}_0^{(2)} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \mathbf{x}^{(2)} = e^{-3t}\left(\mathbf{v}_0^{(2)} + t\mathbf{v}_1^{(2)}\right) = e^{-3t}\left[\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right]$$

For the simple eigenvalue $\lambda_2 = 0$, we find an eigenvector $\mathbf{k}_3 = (4, 4, 1)$, so the general solution is

$$\begin{aligned}\mathbf{x}(t) &= c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{k}_3 \\ &= c_1e^{-3t}\left[\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}\right] + c_2e^{-3t}\left[\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right] + c_3\begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}\end{aligned}$$

□

4.5. *The Phase Plane*

Consider a system of two first order equations

$$x'(t) = f(x,y) \quad \text{and} \quad y'(t) = g(x,y).$$

It is called an autonomous system since f and g are independent of t. So, if $(x(t),y(t))$ is a solution to the system, then the pair $(x(t+c),y(t+c))$ is also a solution to the system for any constant c. Now, let $(x(t),y(t))$ be a solution to the system. If we plot the points $(x(t),y(t))$ on the xy-plane, the resulting curve is called an integral curve of the system, and the xy-plane is called the phase plane.

Let $\mathbf{F}(x,y) = \langle f(x,y), g(x,y)\rangle$ be the vector field on the xy-plane defined by f and g. If we plot the unit vectors defined by those non-zero $\mathbf{F}(x,y)$ at various points of the phase plane, we obtain the direction field of the system. For any point $\mathbf{p}(t) = \langle x(t), y(t)\rangle$ of the system, we have

$$\mathbf{p}'(t) = \langle x'(t), y'(t)\rangle = \langle f(x,y), g(x,y)\rangle = \mathbf{F}(x,y).$$

So, $\mathbf{F}(x,y)$ is everywhere tangential to the integral curve $\mathbf{p}(t)$.

The equation

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$$

is called the phase plane equation. So, any integral curve of the system satisfies the phase plane equation.

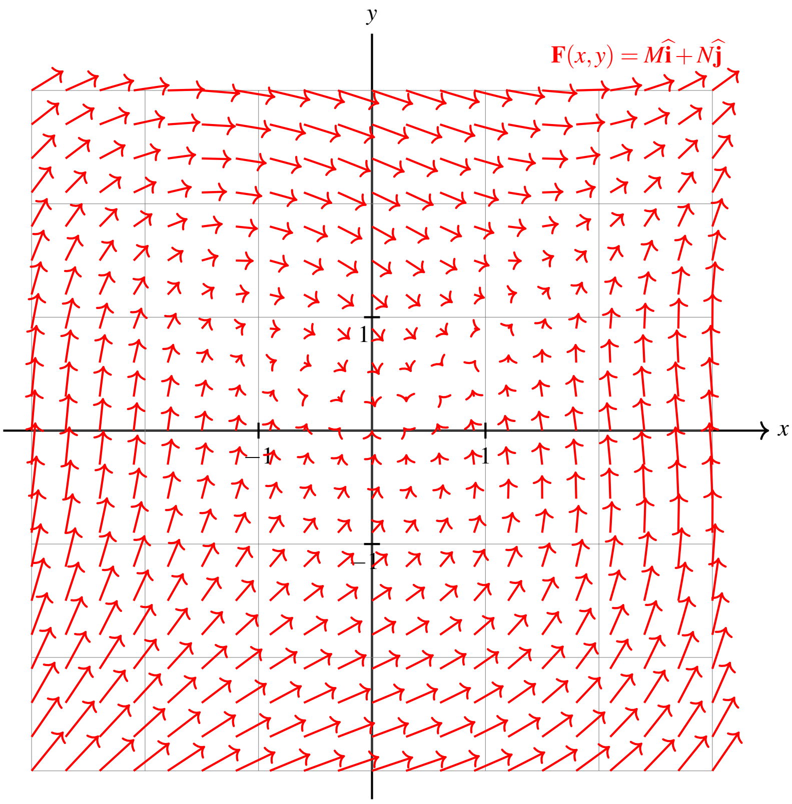

Figure 2: Direction field of $\frac{dx}{dt} = f(x,y)$, $\frac{dy}{dt} = g(x,y)$

A linear autonomous system in the plane has the form

$$x'(t) = a_{11}x + a_{12}y + b_1$$
$$y'(t) = a_{21}x + a_{22}y + b_2$$

where the a_{ij}'s and b_j's are constants. We assume that $a_{11}a_{22} - a_{21}a_{12} \neq 0$. By a simple translation, the system can always be written in the form

$$x'(t) = ax + by$$
$$y'(t) = cx + dy$$

where $ad - bc \neq 0$. The point $(0,0)$ is called a critical point of the system since both $ax + by$ and $cx + dy$ are zero at $x = 0$, $y = 0$. Corresponding to the critical point $(0,0)$ of the system, we have $x(t) = 0$ and $y(t) = 0$ for all t being a constant solution of the system. Our interest is to investigate the behaviour of the integral curves near the critical point $(0,0)$. The behaviour of the solutions to the system is linked to the nature of the roots r_1 and r_2 of the characteristic equation of the system, i.e. r_1 and r_2 are the roots of the quadratic equation

$$X^2 - (a+d)X + (ad - bc) = 0.$$

There are five cases to consider.

(i) **Case 1:** Suppose r_1, r_2 are real and distinct and are of the same sign. Then, the critical point $(0,0)$ is a node. Suppose $r_1 < r_2 < 0$. Then, the general solution to the system has the form

$$x(t) = c_1 A_1 e^{r_1 t} + c_2 A_2 e^{r_2 t}$$
$$y(t) = c_1 B_1 e^{r_1 t} + c_2 B_2 e^{r_2 t}$$

where the A's and B's are fixed constants such that $\frac{B_1}{A_1} \neq \frac{B_2}{A_2}$ and c_1, c_2 are arbitrary constants. When $c_2 = 0$, we have the solution $x(t) = c_1 A_1 e^{r_1 t}$ and $y(t) = c_1 B_1 e^{r_1 t}$. For any $c_1 > 0$, it gives the parametric equation of a half-line $A_1 y = B_1 x$. As $t \to \infty$, the point on this half-line approaches the origin $(0,0)$. For any $c_1 < 0$, it represents the other half of the line $A_1 y = B_1 x$. As $t \to \infty$, the point on this half-line also approaches the origin $(0,0)$.

The two lines $A_1 y = B_1 x$ and $A_2 y = B_2 x$ are called the transformed axes, usually denoted by $\widehat{x}$ and $\widehat{y}$ on the phase plane.

If $c_1 \neq 0$ and $c_2 \neq 0$, the general solution are parametric equations of some curves. Since $r_1 < 0$ and $r_2 < 0$, these curves approach $(0,0)$ as $t \to \infty$. Furthermore,

$$\frac{y}{x} = \frac{c_1 B_1 e^{r_1 t} + c_2 B_2 e^{r_2 t}}{c_1 A_1 e^{r_1 t} + c_2 A_2 e^{r_2 t}} = \frac{(c_1 B_1 / c_2) e^{(r_1 - r_2)t} + B_2}{(c_1 A_1 / c_2) e^{(r_1 - r_2)t} + A_2}.$$

As $r_1 - r_2 < 0$, we see that $\frac{y}{x} \to \frac{B_2}{A_2}$ as $t \to \infty$ so that all the curves enter $(0,0)$ with slope $\frac{B_2}{A_2}$.

A critical point is called a node if it is approached and entered (with a well-defined tangent line) by each integral curve as $t \to \infty$ or $t \to -\infty$. A critical point is said to be stable if for each $R > 0$, there exists a positive $r \leq R$ such that every integral curve which is inside the circle $x^2 + y^2 = r^2$ for some $t = t_0$ remains inside the circle $x^2 + y^2 = R^2$ for all $t > t_0$. Roughly speaking, a critical point is stable if all integral curves that get sufficiently close to the point stay close to the point. If the critical point is not stable, it is called unstable. A critical point is said to be asymptotically stable if it is stable and there exists a circle $x^2 + y^2 = r_0^2$ such that every integral curve which is inside this circle for some $t = t_0$ approaches the critical point as $t \to \infty$. A node is said to be proper if every direction through the node defines an integral curve, otherwise it is said to be improper.

In our situation, we have $(0,0)$ being an asymptotically stable improper node.

If $r_1 > r_2 > 0$, then the situation is exactly the same except that all curves now approach and enter $(0,0)$ as $t \to -\infty$. So all the arrows showing the directions are reversed. The point $(0,0)$ is an unstable improper node.

(ii) **Case 2:** If r_1, r_2 are real, distinct and of opposite sign, then the critical point $(0,0)$ is a saddle point. Let's suppose $r_1 < 0 < r_2$. The general solution is still represented by

$$x(t) = c_1A_1e^{r_1t} + c_2A_2e^{r_2t}$$
$$y(t) = c_1B_1e^{r_1t} + c_2B_2e^{r_2t}$$

The two half-line solutions $x(t) = c_1A_1e^{r_1t}$ and $y(t) = c_1B_1e^{r_1t}$ (for $c_1 > 0$ and $c_1 < 0$) still approach and enter $(0,0)$ as $t \to \infty$, but the other two half-line solutions $x(t) = c_2A_2e^{r_2t}, y(t) = c_2B_2e^{r_2t}$ approach and enter $(0,0)$ as $t \to -\infty$. If $c_1 \neq 0$ and $c_2 \neq 0$, the general solution

$$x(t) = c_1A_1e^{r_1t} + c_2A_2e^{r_2t}$$
$$y(t) = c_1B_1e^{r_1t} + c_2B_2e^{r_2t}$$

defines integral curves of $x'(t) = ax+by$ and $y'(t) = cx+dy$, but since $r_1 < 0 < r_2$, none of these curves approaches $(0,0)$ as $t \to \infty$ or $t \to -\infty$. So $(0,0)$ is not a node. Instead, as $t \to \infty$, each of these curves is asymptotic to one of the half-lines of the line $A_2y = B_2x$; whereas as $t \to -\infty$, each of these curves is asymptotic to one of the half-lines of the line $A_1y = B_1x$. In this case, the critical point is called a saddle point and is certainly unstable.

Example 4.8. Find and classify the critical point of the system

$$x' = 2x+y+3$$
$$y' = -3x-2y-4$$

Solution. Solving $2x+y+3 = 0$ and $-3x-2y-4 = 0$, we have $(x,y) = (-2,1)$. Next, write

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ -4 \end{pmatrix}.$$

The coefficient matrix has eigenvalues ± 1, which are real and of opposite sign. So, the linear system has a saddle point at the origin of the linearised system. ☐

(iii) **Case 3:** If r_1, r_2 are real and equal, then the critical point $(0,0)$ is a node. Let $r_1 = r_2 = r$. In this case, the general solution may or may not involve a factor of t times e^{rt}.

We first consider the subcase where t is not present. Then the general solution is

$$x(t) = c_1e^{rt}$$
$$y(t) = c_2e^{rt}$$

where c_1 and c_2 are arbitrary constants. The integral curves lie on the lines $c_1y = c_2x$. When $r > 0$, the integral curves move away from $(0,0)$, so $(0,0)$ is unstable.

When $r < 0$, the integral curves approach $(0,0)$ and $(0,0)$ is asymptotically stable. In either situation, all integral curves lie on lines passing through $(0,0)$. Because every direction through $(0,0)$ defines an integral curve, the point $(0,0)$ is a proper node.

We now discuss the case where a factor of t times e^{rt} is present. Suppose $r < 0$. The general solution can be written in the form

$$x(t) = c_1 A e^{rt} + c_2(A_1 + At)e^{rt}$$
$$y(t) = c_1 B e^{rt} + c_2(B_1 + Bt)e^{rt}$$

where A's and B's are definite constants and c_1, c_2 are arbitrary constants. When $c_2 = 0$, we obtain the solutions $x(t) = c_1 A e^{rt}$ and $y(t) = c_1 B e^{rt}$. These are solutions representing the two half-lines (for $c_1 > 0$ and $c_1 < 0$) lying on the line $\widehat{y}$ with equation $Ay = Bx$ and slope B/A; and since $r < 0$, both approach $(0,0)$ as $t \to \infty$. Also, since $y/x = B/A$, it is clear that both of these half-lines enter $(0,0)$ with slope B/A.

If $c_2 \neq 0$, the solutions

$$x(t) = c_1 A e^{rt} + c_2(A_1 + At)e^{rt}$$
$$y(t) = c_1 B e^{rt} + c_2(B_1 + Bt)e^{rt}$$

are curves. As $r < 0$, these curves approach $(0,0)$ as $t \to \infty$. Furthermore,

$$\frac{y}{x} = \frac{c_1 B e^{rt} + c_2(B_1 + Bt)e^{rt}}{c_1 A e^{rt} + c_2(A_1 + At)e^{rt}} = \frac{c_1 B/c_2 + B_1 + Bt}{c_1 A/c_2 + A_1 + At}$$

approaches B/A as $t \to \infty$; so these curves all enter $(0,0)$ with slope B/A. We also have $y/x \to B/A$ as $t \to -\infty$. So, each of these curves is tangent to $\hat{y}$ as $t \to \pm\infty$. Consequently, $(0,0)$ is a node that is asymptotically stable.

If $r > 0$, the situation is unchanged except that the directions of the curves are reversed and the critical point is unstable.

(iv) **Case 4:** If r_1, r_2 are conjugate complex but not purely imaginary, then the critical point $(0,0)$ is a spiral.

Let $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$. First, observe that the discriminant of the quadratic equation

$$X^2 - (a+d)X + (ad - bc) = 0$$

is negative. That is,

$$(a+d)^2 - 4(ad - bc) = (a-d)^2 + 4bc < 0.$$

The general solution of the system

$$x'(t) = ax + by$$
$$y'(t) = cx + dy$$

is given by

$$x(t) = e^{\alpha t}[c_1(A_1 \cos\beta t - A_2 \sin\beta t) + c_2(A_1 \sin\beta t + A_2 \cos\beta t)]$$
$$y(t) = e^{\alpha t}[c_1(B_1 \cos\beta t - B_2 \sin\beta t) + c_2(B_1 \sin\beta t + B_2 \cos\beta t)]$$

where A's and B's are definite constants and c's are arbitrary constants.

Suppose $\alpha < 0$. Then, we see that $x \to 0$ and $y \to 0$ as $t \to \infty$. That means all integral curves approach $(0,0)$ as $t \to \infty$. Next we shall show that the integral curves do not enter $(0,0)$ as $t \to \infty$. Instead they wind around like a spiral towards $(0,0)$. To do so, we shall show that the angular coordinate $\theta = \tan^{-1}(y/x)$ is always strictly increasing or strictly decreasing. That is $d\theta/dt > 0$ for all $t > 0$, or $d\theta/dt < 0$ for all

$t > 0$. Differentiating $\theta = \tan^{-1}(y/x)$, we have:

$$\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2+y^2}.$$

Using $x'(t) = ax+by$ and $y'(t) = cx+dy$, we get

$$\frac{d\theta}{dt} = \frac{cx^2+(d-a)xy-by^2}{x^2+y^2}.$$

Since we are interested in solutions that represent integral curves, we assume $x^2+y^2 \neq 0$. Recall that the discriminant of the quadratic equation is < 0, which implies that b and c have opposite signs. We consider the case $b < 0$ and $c > 0$. When $y = 0$, then we have $d\theta/dt = c > 0$. If $y \neq 0$, $d\theta/dt$ cannot be 0. If it were, we have $cx^2+(d-a)xy-by^2 = 0$, or $c(x/y)^2+(d-a)(x/y)-b = 0$ for some real number x/y. But this contradicts the fact that its discriminant is < 0. Thus, we conclude that $d\theta/dt > 0$ for all $t > 0$ when $c > 0$. Similarly in case $b > 0$ and $c < 0$, $d\theta/dt < 0$ for all $t > 0$.

x and y change sign infinitely often as $t \to \infty$, all integral curves must spiral in to $(0,0)$ (counterclockwise or clockwise according to $c > 0$ or $c < 0$). The critical point in this case is a spiral, and is asymptotically stable.

If $\alpha > 0$, the situation is the same except that the integral curves approach $(0,0)$ as $t \to -\infty$ and the critical point is unstable.

(v) **Case 5:** If r_1, r_2 are purely imaginary, then the critical point $(0,0)$ is a centre.

The general solution is given by

$$\begin{aligned} x(t) &= c_1(A_1\cos\beta t - A_2\sin\beta t) + c_2(A_1\sin\beta t + A_2\cos\beta t)\\ y(t) &= c_1(B_1\cos\beta t - B_2\sin\beta t) + c_2(B_1\sin\beta t + B_2\cos\beta t)\end{aligned}$$

Thus $x(t)$ and $y(t)$ are periodic functions of period 2π so that each integral curve is a closed path surrounding the origin. These curves can be shown to be ellipses by solving the phase plane equation

$$\frac{dy}{dx} = \frac{cx+dy}{ax+by}.$$

Example 4.9. Show that the integral curves of the system

$$\begin{aligned} x' &= -\beta y\\ y' &= \beta x\end{aligned}$$

are circles.

Solution. Differentiating the first equation yields $x'' = -\beta y' = -\beta^2 x$. The characteristic equation is $m^2 + \beta^2 = 0$, which has roots $m = \pm\beta i$. Hence, the solution is

$$x(t) = A\cos(\beta t) + B\sin(\beta t).$$

Hence, $y(t) = A\sin(\beta t) - B\cos(\beta t)$. One can show that $x^2+y^2 = A^2+B^2$, which represents a circle centred at the origin of radius $\sqrt{A^2+B^2}$. □

We now summarise the above discussion.

Theorem 4.9 (classification threm). Assume $(0,0)$ is an isolated critical point of the linear system $x' = ax+by$ and $y'cx+dy$, where a,b,c,d are real and $ad-bc \neq 0$. Let r_1, r_2 be the roots of the characteristic equation

$$X^2-(a+d)X+(ad-bc)=0.$$

The stability of the origin and the classification of the origin as a critical point depends on the roots r_1, r_2 as follows:

Roots	Type of critical point	Stability
distinct, positive	improper node	unstable
distinct, negative	improper node	asymptotically stable
opposite signs	saddle point	unstable
equal, positive	proper or improper node	unstable
equal, negative	proper or improper node	asymptotically stable
complex value with positive real part	spiral point	unstable
complex value with negative real part	spiral point	asymptotically stable
purely imaginary	centre	stable

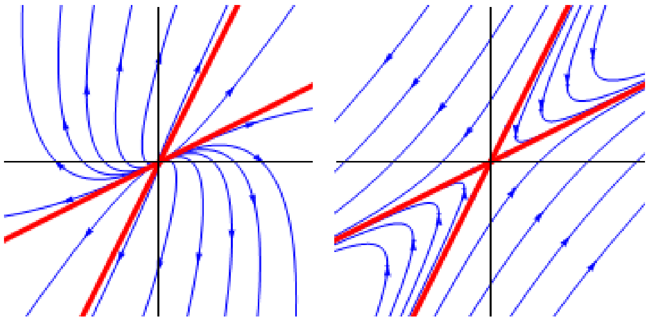

Figure 3: Unstable node and saddle

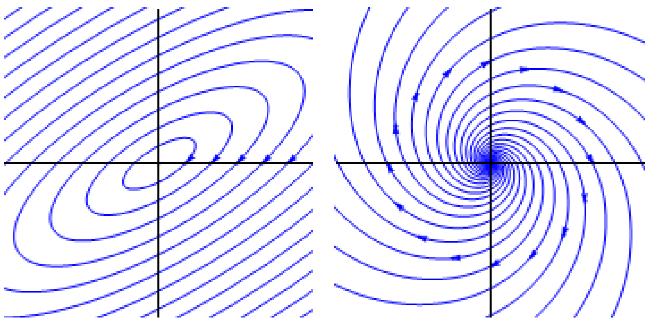

Figure 4: Centre and spiral

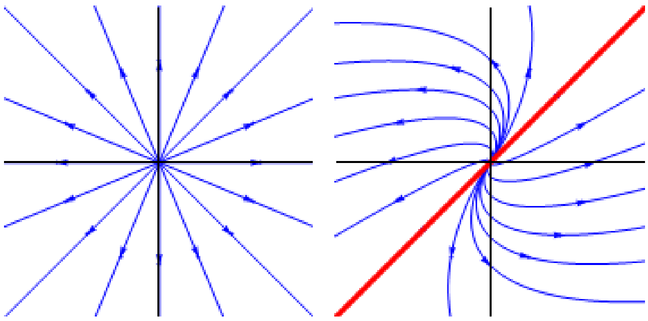

Figure 5: Star and improper node

5. Power Series Solutions

5.1. *Power Series*

We do a bit of recap of MA3210.

Definition 5.1 (power series)**.** An infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1 (x-x_0) + a_2 (x-x_0)^2 + \dots \quad \text{is a power series in } x-x_0.$$

In this section, we will pay close attention to the point $x_0 = 0$. Substituting $x_0 = 0$ into the infinite series in Definition 5.1, we obtain

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots.$$

This series is said to converge at a point x if

$$\lim_{m\to\infty} \sum_{n=0}^{m} a_n x^n \quad \text{exists,}$$

and the value of this series is the value of this limit. Recall that each power series like

$$\sum_{n=0}^{\infty} a_n x^n \quad \text{corresponds to a positive real number } R \quad \text{called the radius of convergence.}$$

The radius of convergence has the following property:

$$\text{the series converges if } |x| < R \quad \text{and} \quad \text{diverges if } |x| > R.$$

We say that $R = 0$ when the series converges only at $x = 0$, and equal to ∞ when it converges for all x. In many important cases, R can be obtained by the ratio test (Theorem 5.1).

Theorem 5.1 (ratio test)**.** Consider the series

$$\sum_{n=0}^{\infty} a_n x^n.$$

If each $a_n \neq 0$ and if for a fixed point $x \neq 0$ we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x| = L,$$

then

$$\text{the series converges for } L < 1 \quad \text{and} \quad \text{diverges for } L > 1.$$

As such,

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \quad \text{if the limit exists.}$$

If the limit of the ratio is infinity, then we say that $R = \infty$.

Definition 5.2 (interval of convergence). The interval $(-R,R)$ is called the interval of convergence in the sense that inside the interval, the series converges. Outside the interval, the series diverges.

Example 5.1. The power series

$$\sum_{n=0}^{\infty} n!x^n = 1+x+2!x^2+3!x^3+\dots$$

converges only at $x=0$, so $R=0$.

Example 5.2. The power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$$

converges for all x, so $R=\infty$. In fact, this is the famous Maclaurin series for e^x.

Example 5.3. The power series

$$\sum_{n=0}^{\infty} x^n = 1+x+x^2+\dots$$

converges for $|x|<1$, so $R=1$. In fact, this is a geometric series with first term 1 and common ratio x.

Proposition 5.1. Suppose the series

$$\sum_{n=0}^{\infty} a_n x^n \quad \text{converges for } |x|<R \text{ with } R>0.$$

We denote its sum by $f(x)$. That is,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

so one can prove that f is continuous and has derivatives of all orders for $|x|<R$. Also, the series can be differentiated termwise in the sense that

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 3\cdot 2a_3 x + \dots$$

and so on. The resulting series are still convergent for $|x|<R$. These successive differentiated series yield the following basic formula relating a_n with $f(x)$ and its derivatives. That is,

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

Proposition 5.2. Suppose the series

$$\sum_{n=0}^{\infty} a_n x^n \quad \text{converges for } |x|<R \text{ with } R>0.$$

The series can be integrated termwise provided that the limits of integration lie inside the interval of convergence.

Proposition 5.3. Suppose

$$f(x)=\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+\dots \quad \text{and} \quad g(x)=\sum_{n=0}^{\infty}b_nx^n=b_0+b_1x+b_2x^2+\dots$$

are power series with interval of convergence $|x|<R$. Then, the series can be added or subtracted termwise, i.e.

$$f(x)\pm g(x)=\sum_{n=0}^{\infty}(a_n\pm b_n)x^n=(a_0\pm b_0)+(a_1\pm b_1)x+(a_2\pm b_2)x^2+\dots$$

Just like multiplication of polynomials (convolution), we have

$$f(x)g(x)=\sum_{n=0}^{\infty}c_nx^n \quad \text{where} \quad c_n=a_0b_n+a_1b_{n-1}+\dots+a_nb_0.$$

Theorem 5.2. Suppose two power series

$$f(x)=\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+\dots \quad \text{and} \quad g(x)=\sum_{n=0}^{\infty}b_nx^n=b_0+b_1x+b_2x^2+\dots$$

converge to the same function so that $f(x)=g(x)$ for $|x|<R$. Then, because $a_n=f^{(n)}(0)/n!$, we have $a_n=b_n$ for all n. In particular, if

$$f(x)=0 \text{ for all } |x|<R \quad \text{then} \quad a_n=0 \text{ for all } n\in\mathbb{Z}_{\geq 0}.$$

Let $f(x)$ be a continuous function that has derivatives of all orders for $|x|<R$. Can it be represented by a power series? Given that $a_n=f^{(n)}(0)/n!$, it is natural to expect

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\dots \quad \text{for all } |x|<R.$$

Unfortunately, this is not always true. Instead, one can use Taylor's expansion for $f(x)$†, which is

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k+R_n(x) \quad \text{where} \quad \text{the remainder } R_n(x)=\frac{f^{(n+1)}(x_0)}{(n+1)!}x^{n+1}$$

for some $x_0\in(0,x)$.

Example 5.4. The following familiar expansions are valid for all x:

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$$
$$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}+\dots$$
$$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}+\dots$$

†This is formally called Taylor's theorem with Lagrange form of the remainder.

Definition 5.3 (analytic function). A function $f(x)$ with the property that a power series expansion of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is valid in some interval containing the point x_0 is said to be analytic at x_0. In this case,

$$a_n = \frac{f^{(n)}(x_0)}{n!} \quad \text{and} \quad \text{the above series is called the Taylor series of } f(x) \text{ at } x_0.$$

One would encounter analytic functions (Definition 5.3) in MA3211 Complex Analysis.

Example 5.5. e^x, $\sin x$ and $\cos x$ are analytic for all $x \in \mathbb{R}$.

Proposition 5.4. The following hold for analytic functions:

(i) Polynomials, e^x, $\sin x$ and $\cos x$ are analytic at all points

(ii) If $f(x)$ and $g(x)$ are analytic at x_0, then

$$f(x) \pm g(x), f(x)g(x), f(x)/g(x) \quad \text{are analytic at } x_0$$

For the quotient $f(x)/g(x)$, we further add the constraint that $g(x_0) \neq 0$

(iii) In relation to the inverse function theorem, if $f(x)$ is analytic at x_0 and $f^{-1}(x)$ is continuous inverse, then

$$f^{-1}(x) \text{ is analytic at } f(x_0) \quad \text{if} \quad f'(x_0) \neq 0$$

(iv) If $g(x)$ is analytic at x_0 and $f(x)$ is analytic at $g(x_0)$, then $f(g(x))$ is analytic at x_0

(v) The sum of power series is analytic at all points inside the interval of convergence

5.2. *Series Solutions of First-Order Solutions*

A first-order ODE $y' = f(x,y)$ can be solved by assuming it has a power series solution. Here are two familiar examples.

Example 5.6. Consider the differential equation

$$y' = y.$$

We assume that it has a power series solution of the form

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots \quad \text{which converges for } |x| < R.$$

So, the equation $y' = y$ has a solution which is analytic at the origin. Then,

$$y' = a_1 + 2a_2 x + \ldots + n a_n x^{n-1} \quad \text{which converges for } |x| < R.$$

Since $y' = y$, both series have the same coefficients. As such, we obtain the recurrence relation

$$(n+1) a_{n+1} = a_n \quad \text{for } n = 0, 1, 2, \ldots$$

so by recursion, we have

$$a_n = \frac{1}{n!} a_0.$$

As such,

$$y = a_0 \left(1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots \right)$$

where a_0 is an arbitrary constant. Of course, this is the Maclaurin expansion of e^x, which has the general solution $y = a_0 e^x$.

Example 5.7. The function $y = (1+x)^p$ where $p \in \mathbb{R}$ satisfies the differential equation

$$(1+x)\,y' = py \quad \text{where} \quad y(0) = 1.$$

Just like before, we assume that y has a power series solution of the form

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots \quad \text{with positive radius of convergence.}$$

Then,

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + (n+1)\,a_{n+1} x^n + \ldots$$

so

$$xy' = a_1 x + 2a_2 x^2 + \ldots + na_n x^n + \ldots \quad \text{and} \quad py = pa_0 + pa_1 x + pa_2 x^2 + \ldots + pa_n x^n + \ldots$$

By considering the differential equation and equating coefficients, we have

$$(n+1)\,a_{n+1} + na_n = pa_n \quad \text{so} \quad a_{n+1} = \frac{p-n}{n+1} a_n.$$

This recurrence relation holds for all $n \in \mathbb{Z}_{\geq 0}$. So,

$$\begin{aligned} a_1 &= p \\ a_2 &= \frac{p(p-1)}{2} \\ a_3 &= \frac{p(p-1)(p-2)}{2 \cdot 3} \end{aligned}$$

so in general, one can deduce that

$$a_n = \frac{p(p-1)\ldots(p-n+1)}{n!}.$$

In other words,

$$y = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{2 \cdot 3} x^3 + \ldots + \frac{p(p-1)\ldots(p-n+1)}{n!} x^n + \ldots$$

for which one can deduce that the series converges for $|x| < 1$ by the ratio test. Since the differential equation has a unique solution (due to the initial value), we conclude that

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{2 \cdot 3} x^3 + \ldots + \frac{p(p-1)\ldots(p-n+1)}{n!} x^n + \ldots \quad \text{for } |x| < 1.$$

This is in fact the binomial expansion of $(1+x)^p$.

5.3. *Second-Order Linear Equations and Ordinary Points*

Definition 5.4 (ordinary and singular points). Consider the homogeneous second-order linear ODE

$$y'' + P(x)y' + Q(x)y = 0.$$

The point x_0 is an ordinary point of the ODE if $P(x)$ and $Q(x)$ are analytic at x_0. If at $x = x_0$, $P(x)$ and/or $Q(x)$ are not analytic, then x_0 is a singular point of the ODE.

Definition 5.5 (singular point). A singular point x_0 at which the functions

$$(x - x_0)P(x) \text{ and } (x - x_0)^2 Q(x) \quad \text{are analytic}$$

is a regular singular point of the ODE $y'' + P(x)y' + Q(x)y = 0$. If a singular point x_0 is not a regular singular point, then it is an irregular singular point.

Example 5.8. If $P(x)$ and $Q(x)$ are constant functions, then every point is an ordinary point of $y'' + P(x)y' + Q(x)y = 0$.

Example 5.9. For the differential equation $y'' + xy = 0$, the function $Q(x) = x$ is analytic at every point, so every point is an ordinary point.

Example 5.10. Consider the Cauchy-Euler equation

$$y'' + \frac{a_1}{x}y' + \frac{a_2}{x^2}y = 0 \quad \text{where } a_1 \text{ and } a_2 \text{ are constants.}$$

Then, $x = 0$ is a singular point but every other point is an ordinary point.

Example 5.11. Consider the differential equation

$$y'' + \frac{1}{(x-1)^2}y' + \frac{8}{x(x-1)}y = 0.$$

The singular points are 0 and 1. At the point 0,

$$xP(x) = \frac{x}{(x-1)^2} \text{ and } x^2 Q(x) = -\frac{8x}{x-1} \quad \text{are analytic at } x = 0.$$

So, 0 is a regular singular point. At the point 1, $(x-1)P(x) = 1/(x-1)$ which is not analytic at $x = 1$, so $x = 1$ is an irregular singular point.

To discuss the behaviour of the singularities at infinity, naturally, we use the substitution $x = 1/t$. This converts the problem to the behaviour of the transformed equation near the origin. As such, the original differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \quad \text{becomes} \quad \frac{d^2y}{dt^2} + \left(\frac{2}{t} - \frac{1}{t^2}P\left(\frac{1}{t}\right)\right)\frac{dy}{dt} + \frac{1}{t^4}Q\left(\frac{1}{t}\right)y = 0.$$

Hopefully, the reader would know how to define the point at infinity.

Example 5.12. Consider the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{2}\left(\frac{1}{x^2} + \frac{1}{x}\right)\frac{dy}{dx} + \frac{1}{2x^3}y = 0.$$

The substitution $x = 1/t$ transforms the differential equation into

$$\frac{d^2y}{dt^2} + \left(\frac{3-t}{2t}\right)\frac{dy}{dt} + \frac{1}{2t}y = 0.$$

As $t = 0$ is a regular singular point of the new differential equation, we say that the point at inifinity is a regular singular point of the original differential equation.

Example 5.13 (hypergeometric equation)**.** Consider the hypergeometric differential equation

$$x(1-x)\frac{d^2y}{dx^2}+[c-(a+b+1)x]\frac{dy}{dx}-aby=0 \quad \text{where } a,b,c \text{ are constants.}$$

Show that the equation has precisely 3 regular singular points at $0,1,\infty$.

Solution. We have

$$\frac{d^2y}{dx^2}+\frac{c-(a+b+1)x}{x(1-x)}\frac{dy}{dx}-\frac{ab}{x(1-x)}y=0.$$

So,

$$P(x)=\frac{c-(a+b+1)x}{x(1-x)} \quad \text{and} \quad Q(x)=-\frac{ab}{x(1-x)}.$$

One checks that $x=0$ and $x=1$ are regular singular points. By using the substitution $t=1/x$, one can deduce that $t=0$ is a regular singular point, so $x=\infty$ is a regular singular point. □

Theorem 5.3. Let x_0 be an ordinary point of the differential equation

$$y''+P(x)y'+Q(x)y=0 \quad \text{where } a_0 \text{ and } a_1 \text{ are constants.}$$

Then, there exists a unique function $y(x)$ that is analytic at x_0 which is a solution to the differential equation in an interval containing x_0, and satisfies the initial conditions $y(x_0)=a_0$ and $y'(x_0)=a_1$.

Moreover, if the power series expansions of $P(x)$ and $Q(x)$ are valid on an interval $|x-x_0|<R$, where $R>0$, then the power series expansion of this solution is also valid on the same interval.

Definition 5.6 (Legendre's equation)**.** The differential equation

$$\left(1-x^2\right)y''-2xy'+p(p+1)y=0$$

is known as Legendre's equation. Here, p is a constant called the order of the equation.

The functions defined in the series solution of Legendre's equation are called Legendre functions. When $p\in\mathbb{Z}_{\geq 0}$, one of these series terminates and becomes a polynomial in x. For example, if $p=n$ is a positive even integer, the series representing y_1 terminates and y_1 is a polynomial of degree n; if $p=n$ is odd, y_2 is again a polynomial of degree n. These are called Legendre polynomials $P_n(x)$ and they give particular solutions to Legendre's equation

$$\left(1-x^2\right)y''-2xy'+n(n+1)y=0 \quad \text{where } n\in\mathbb{Z}_{\geq 0}.$$

We define the first six Legendre polynomials as follows (please refer to the graphs of $P_0(x),\ldots,P_4(x)$ in Figure 6):

$$\begin{aligned}
P_0(x)&=1\\
P_1(x)&=x\\
P_2(x)&=\frac{1}{2}\left(3x^2-1\right)\\
P_3(x)&=\frac{1}{2}\left(5x^3-3x\right)\\
P_4(x)&=\frac{1}{8}\left(35x^4-30x^2+3\right)\\
P_5(x)&=\frac{1}{8}\left(63x^5-70x^3+15x\right)
\end{aligned}$$

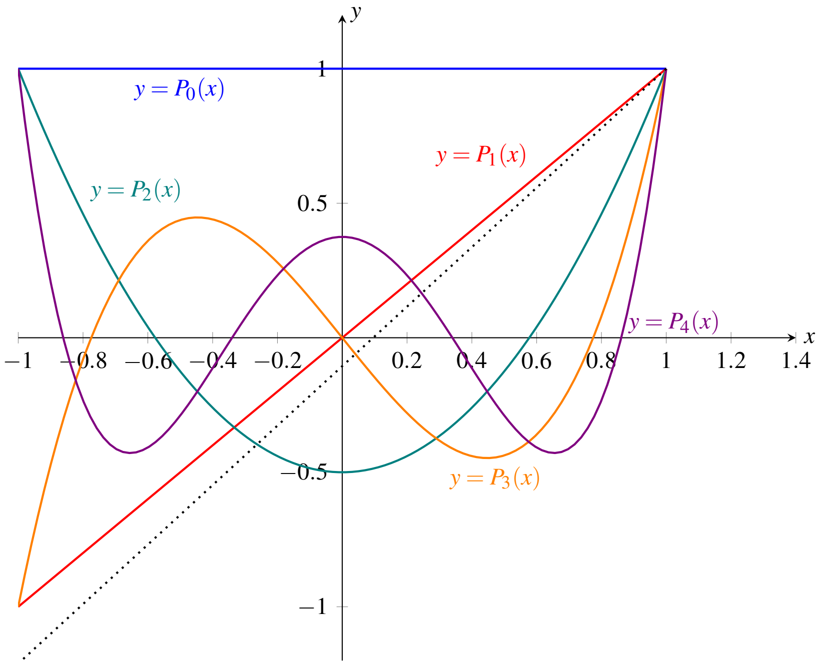

Figure 6: The graphs of the first five Legendre polynomials $P_n(x)$ where $0 \leq n \leq 4$

Returning to Legendre's differential equation, we have

$$P(x) = -\frac{2x}{1-x^2} \quad \text{and} \quad Q(x) = \frac{p(p+1)}{1-x^2}.$$

The origin is an ordinary point, and

$$\text{we expect} \quad \text{a solution of the form } y = \sum_{n=0}^{\infty} a_n x^n.$$

As such, the differential equation can be written as

$$\left(1-x^2\right)\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - 2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + p(p+1)\sum_{n=0}^{\infty}a_n x^n = 0.$$

Equivalently,

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=2}^{\infty}n(n-1)a_n x^n - \sum_{n=1}^{\infty}2na_n x^n + \sum_{n=0}^{\infty}p(p+1)a_n x^n = 0.$$

Since each series contains terms in x^n and the sum of these series is zero as mentioned, the coefficients of x^n must be zero for all n. This yields the second-order recurrence relation

$$(n+1)(n+2)a_{n+2} - n(n-1)a_n - 2na_n + p(p+1)a_n = 0 \quad \text{for all } n \geq 2.$$

Equivalently, the recurrence relation can also be written as

$$a_{n+2} = -\frac{(p-n)(p+n+1)}{(n+1)(n+2)}a_n.$$

We can consider n being odd and n being even, so that we can obtain an explicit expression of the recurrence relation in terms of a_1 and a_0 respectively. We omit the details here, but anyway one can check that the desired explicit expression for the n^{th} Legendre polynomial $P_n(x)$ is

$$P_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}.$$

Rodrigues' formula (Theorem 5.4) provides a nice formula for generating the n^{th} Legendre polynomial.

Theorem 5.4 (Rodrigues' formula)**.** We have

$$P_n(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n} \left(x^2-1\right)^n.$$

Definition 5.7 (Hermite's equation)**.** Let p be a constant. Then,

$$y'' - 2xy' + 2py = 0$$

is known as Hermite's equation.

We leave it to the reader to verify that the general solution to Hermite's equation is $y(x) = a_0 y_1(x) + a_1 y_2(x)$, where

$$\begin{aligned} y_1(x) &= 1 - \frac{2p}{2!}x^2 + \frac{2^2(p-2)}{4!}x^4 - \frac{2^3 p(p-2)(p-4)}{6!}x^6 + \dots \\ y_2(x) &= x - \frac{2(p-1)}{3!}x^3 + \frac{2^2(p-1)(p-3)}{5!}x^5 - \frac{2^3(p-1)(p-3)(p-5)}{7!}x^7 + \dots \end{aligned}$$

The Hermite polynomial of degree n, denoted by $H_n(x)$, is the n^{th} degree polynomial solution to Hermite's equation, multiplied by a suitable constant so that the coefficient of x^n is 2^n. The first six Hermite polynomials are

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= 2x \\ H_2(x) &= 4x^2 - 2 \\ H_3(x) &= 8x^3 - 12x \\ H_4(x) &= 16x^4 - 48x^2 + 12 \\ H_5(x) &= 32x^5 - 160x^3 + 120x \end{aligned}$$

5.4. *Regular Singular Points and Frobenius' Method*

Consider the second order linear homogeneous differential equation

$$x^2 y'' + xp(x)y' + q(x)y = 0,$$

where $p(x)$ and $q(x)$ are analytic at $x = 0$. In other words, 0 is said to be a regular singular point of the differential equation. We write $p(x)$ and $q(x)$ as power series, i.e.

$$p(x) = p_0 + p_1 x + p_2 x^2 + \dots \quad \text{and} \quad q(x) = q_0 + q_1 x + q_2 x^2 + \dots.$$

Suppose the differential equation has a series solution of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

An infinite series of this form is called a Frobenius series, and the method that we will be using to solve the differential equation is known as the method of Frobenius. We may assume that $a_0 \neq 0$ because the series must have a first non-zero term. Termwise differentiation yields

$$y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}.$$

One can substitute the series of y, y', y'' into the differential equation. Thereafter, we omit the algebraic manipulation but the idea is that the coefficient $r(r-1)a_0 + p_0 r a_0 + q_0 a_0$ of x^r must vanish. Since $a_0 \neq 0$, it follows that r must satisfy the quadratic equation

$$r(r-1) + p_0 r + q_0 = 0.$$

We call this the indicial equation, which is the same equation obtained with the Cauchy-Euler equation. The two roots of the indicial equation (which are possibly equal) are the exponents of the differential equation at the regular singular point $x = 0$.

Let r_1 and r_2 be the roots of the indicial equation. If $r_1 \neq r_2$, then there are two possible Frobenius solutions and they are linearly independent. On the other hand, if $r_1 = r_2$, then there is only one possible Frobenius series solution. The second one cannot be a Frobenius series and can only be found by other means.

Example 5.14. Find the exponents in the possible Frobenius series solutions of the differential equation

$$2x^2(1+x)y'' + 3x(1+x)^3 y' - (1-x^2)y = 0.$$

Solution. Note that $x = 0$ is a regular singular point since $p(x) = \frac{3}{2}(1+x)^2$ and $q(x) = -\frac{1}{2}(1-x)$ are polynomials. Rewriting the equation in standard form yields

$$y'' + \frac{\frac{3}{2}(1+2x+x^2)}{x}y' + \frac{-\frac{1}{2}(1-x)}{x^2}y = 0.$$

We see that $p_0 = \frac{3}{2}$ and $q_0 = -\frac{1}{2}$, so the indicial equation is

$$r(r-1) + \frac{3}{2}r - \frac{1}{2} = 0.$$

The roots are $r_1 = \frac{1}{2}$ and $r_2 = -1$, so the two possible Frobenius series soliutions are of the form

$$y_1(x) = x^{1/2}\sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{-1}\sum_{n=0}^{\infty} a_n x^n.$$

□

Example 5.15. Find the Frobenius series solutions of

$$xy'' + 2y' + xy = 0.$$

Solution. Rewrite the equation in standard form so we obtain

$$x^2 y'' + 2xy' + x^2 y = 0.$$

So, $p(x) = 2$ and $q(x) = x^2$. Thus, $p_0 = 2$ and $q_0 = 0$ and the indicial equation is $r(r-1) + 2r = 0$. Hence, $r_1 = 0$ and $r_2 = -1$. In this case, $r_1 - r_2$ is an integer and we may not have two Frobenius series solutions. Having said that, we know that there is a Frobenius series solution corresponding to $r_1 = 0$. We consider the possibility of the solution corresponding to the smaller exponent $r_2 = -1$. We begin with

$$y = x^{-1}\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n-1}.$$

Substituting into the given equation, we obtain

$$\sum_{n=0}^{\infty}(n-1)(n-2)c_n x^{n-2} + 2\sum_{n=0}^{\infty}(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0.$$

Hence,

$$\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=2}^{\infty} c_{n-2}x^{n-2} = 0.$$

Equating coefficients, we obtain the recurrence relation

$$c_n = -\frac{c_{n-2}}{n(n-1)} \quad \text{for } n \geq 2.$$

It follows that for $n \geq 1$, we have

$$c_{2n} = \frac{(-1)^n c_0}{(2n)!} \quad \text{and} \quad c_{2n+1} = \frac{(-1)^n c_1}{(2n+1)!}.$$

Hence,

$$y = x^{-1}\sum_{n=0}^{\infty} c_n x^n = \frac{c_0}{x}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} + \frac{c_1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}.$$

By recalling some classic series expansions, we recognise this general solution as $y = \frac{1}{x}(c_0\cos x + c_1 \sin x)$. On the other hand, if we begin with the larger exponent, we will obtain the solution $\frac{\sin x}{x}$. □

Example 5.16 (MA3220 AY14/15 Sem 1 Tutorial 10)**.** Find the Frobenius series solutions of the differential equation

$$xy'' + 2y' + 9xy = 0 \quad \text{where } x > 0.$$

Solution. Assume that the differential equation has a series solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

Then,

$$y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}.$$

Substituting all these into the differential equation yields

$$\sum_{n=0}^{\infty} a_n\left[(n+r)(n+r-1) + 2(n+r)\right]x^{n+r-1} + \sum_{n=0}^{\infty} 9a_n x^{n+r+1} = 0.$$

We then look at the lowest power term which is x^{r-1}. Setting $n = 0$ yields $r(r-1)+2r = 0$, which is our indicial equation. It has roots $r_1 = -1$ and $r_2 = 0$. When $r = 0$, substituting this into our series yields

$$\sum_{n=0}^{\infty} n(n+1)a_n x^{n-1} + \sum_{n=0}^{\infty} 9a_n x^{n+1} = 0.$$

We then combine this into a single series, so

$$\sum_{n=2}^{\infty}\left[n(n+1)a_n + 9a_{n-2}\right]x^{n-1} = 0 \quad \text{so} \quad n(n+1)a_n + 9a_{n-2} = 0.$$

The recurrence relation can be rewritten as

$$a_n = -\frac{9}{n(n+1)}a_{n-2}.$$

By repeatedly applying the recurrence relation, we have

$$y_1(x) = a_0\left(1 - \frac{9}{2\cdot 3}x^2 + \frac{9^2}{2\cdot 3\cdot 4\cdot 5}x^4 - \cdots\right) = a_0\sum_{n=0}^{\infty}\frac{(-1)^n\,9^n}{(2n+1)!}x^{2n} = a_0\cdot\frac{\cos 3x}{x}.$$

When $r = -1$, one can obtain the Frobenius series solution being some constant times $\frac{\sin 3x}{3x}$. As such, we have obtained two linearly independent solutions. $\square$

6. Fundamental Theory of Ordinary Differential Equations

6.1. *Existence-Uniqueness Theorems*

Here, we consider the initial value problem

$$\frac{dx}{dt} = f(t,x) \quad \text{where } x(t_0) = x_0.$$

Definition 6.1 (Lipschitz continuity)**.** Let $G \subseteq \mathbb{R}^2$. We say that $f(t,x) : G \to \mathbb{R}^2$ satisfies a Lipschitz condition with respect to x in G if there exists a constant $L > 0$ such that for any $(t,x_1),(t,x_2) \in G$,

$$|f(t,x_1) - f(t,x_2)| \leq L\,|x_1 - x_2|\,.$$

L is called a Lipschitz constant.

Theorem 6.1 (Picard)**.** Let $f(t,x)$ be continuous on the rectangle

$$R : |t - t_0| \leq a \text{ and } |x - x_0| \leq b \quad \text{where } a, b > 0.$$

Also, let $|f(t,x)| \leq M$ for all $(t,x) \in R$. Furthermore, assume f satisfies a Lipschitz condition with constant L in R. Then, there exists a unique solution to the initial value problem

$$\frac{dx}{dt} = f(t,x) \text{ with } x(t_0) = x_0 \quad \text{on the interval } I = [t_0 - \alpha, t_0 + \alpha] \text{ where } \alpha = \min\left\{a, \frac{b}{M}\right\}.$$

Example 6.1. Let $f(t,x) = x^2 e^{-t^2} \sin t$ be defined on

$$G = \left\{(t,x) \in \mathbb{R}^2 : 0 \leq x \leq 2\right\}.$$

Let $(t,x_1),(t,x_2) \in G$. Then,

$$|f(t,x_1) - f(t,x_2)| = \left|x_1^2 e^{-t^2} \sin t - x_2^2 e^{-t^2} \sin t\right| = \left|e^{-t^2} \sin t\right| |x_1 + x_2|\, |x_1 - x_2|$$

Based on the graph of $y = \left|e^{-t^2} \sin t\right|$, we see that the maximum value is $\approx 0.397 \leq 1$, and $|x_1 + x_2| \leq 4$, so we may take $L = 1 \cdot 4$ so f satisfies a Lipschitz condition on G with Lipschitz constant 4.

Example 6.2. Let $f(t,x) = t\sqrt{x}$ be defined on

$$G = \left\{(t,x) \in \mathbb{R}^2 : 0 \leq t \leq 1, 0 \leq x \leq 1\right\}.$$

Consider the two points $(1,x),(1,0) \in G$. We have

$$|f(1,x) - f(1,0)| = \sqrt{x} = \frac{1}{\sqrt{x}}\,|x - 0|\,.$$

However, as $x \to 0^+$, we have $\frac{1}{\sqrt{x}} \to +\infty$, so f cannot satisfy the Lipschitz condition on G for any finite constant $L > 0$.

Proposition 6.1. Suppose $f(t,x)$ has a continuous partial derivative $f_x(t,x)$ on a rectangle R on the tx-plane. Then, f satisfies a Lipschitz condition on R.

Example 6.3. Let $f(t,x) = x^2$ be defined on $G = \left\{(t,x) \in \mathbb{R}^2 : 0 \leq t \leq 1\right\}$. Then,

$$|f(t,x_1) - f(t,x_2)| = |x_1 + x_2|\,|x_1 - x_2|\,.$$

Since x_1 and x_2 can be arbitrarily large, then f cannot satisfy a Lipschitz condition on G. However, if we replace G with any compact subset of $\mathbb{R}^2$ (equivalent to closed and bounded by the Heine-Borel theorem), then f will satisfy the Lipschitz condition.

Example 6.4 (MA3220 AY14/15 Sem 1 Tutorial 11)**.** Let

$$R = \left\{(t,x) \in \mathbb{R}^2 : |t| \le a, |x| \le b\right\} \quad \text{where } a,b > 0.$$

Let $f(t,x) = t\sin x + x\cos t$ be defined on R. Show that f satisfies a Lipschitz condition with respect to x on R.

Solution. We have

$$\begin{aligned} |f(t,x_1) - f(t,x_2)| &= |t\sin x_1 - t\sin x_2 + x_1\cos t - x_2\cos t| \\ &\le |t|\,|\sin x_1 - \sin x_2| + |\cos t|\,|x_1 - x_2| \\ &\le a \cdot 2 + 1 \cdot 2b = 2a + 2b \end{aligned}$$

so we can take the Lipschitz constant to be $2a+2b$. Hence, f satisfies the Lipschitz condition with respect to x on R. □

6.2. *The Method of Successive Approximations*

6.3. *Gronwall's Inequality*

Theorem 6.2. Let $f, g, h \ge 0$ be continuous functions defined for $t \ge t_0$. If

$$f(t) \le h(t) + \int_{t_0}^{t} g(s)f(s)\,ds \text{ for } t \ge t_0 \quad \text{then} \quad f(t) \le h(t) + \int_{t_0}^{t} g(s)h(s)e^{\int_s^t g(u)\,du}\,ds \text{ for } t \ge t_0.$$

Proof. Let

$$z(t) = \int_{t_0}^{t} g(s)f(s)\,ds.$$

Then, for $t \ge t_0$, we have $z'(t) = g(t)f(t)$ by the fundamental theorem of Calculus. Since $g(t) \ge 0$, multiplying both sides of the given inequality by $g(t)$ yields

$$z'(t) \le g(t)[h(t) + z(t)] \quad \text{so} \quad z'(t) - g(t)z(t) \le g(t)h(t).$$

This is a first order differential inequality which can be easily solved by the integrating factor method. We omit the remaining details. □

Theorem 6.3 (Gronwall's inequality)**.** Let $f, g \ge 0$ be continuous functions for $t \ge t_0$. Also, let k be any non-negative constant. If

$$f(t) \le k + \int_{t_0}^{t} g(s)f(s)\,ds \text{ for } t \ge t_0 \quad \text{then} \quad f(t) \le ke^{\int_{t_0}^t g(s)\,ds} \text{ for } t \ge t_0.$$

Proof. Set $h(t) = k$ in Theorem 6.2. □

Corollary 6.1. Let f be a continuous non-negative function for $t \ge t_0$ and k be a non-negative constant. If

$$f(t) \le k\int_{t_0}^{t} f(s)\,ds \text{ for all } t \ge t_0 \quad \text{then} \quad f(t) = 0 \text{ for all } t \ge t_0.$$

Corollary 6.2. Let $f(t,x)$ be a continuous function which satisfies a Lipschitz condition on R with a Lipschitz constant L, where R is either a rectangle or a strip. If ϕ and φ are two solutions to the initial value problem

$$x' = f(t,x) \quad \text{where } x(t_0) = x_0,$$

on an interval I containing t_0, then $\phi(t) = \varphi(t)$ for all $t \in I$.

Theorem 6.4 (Peano). Suppose $G \subseteq \mathbb{R}^2$ is an open subset containing (t_0, x_0) and $f(t,x)$ is continuous on G. Then, there exists $a > 0$ such that the initial value problem has at least one solution on $[t_0 - a, t_0 + a]$.

Example 6.5. Suppose $\phi(t)$ is a solution to the initial value problem

$$\frac{dx}{dt} = \frac{x^3 - x}{1 + t^2x^2} \quad \text{where } x(0) = \frac{1}{2}.$$

Show that $0 < \phi(t) < 1$ for all $t \in J$, where $\phi(t)$ is defined on the open interval J containing 0.

Solution. Let $\phi(t)$ be a solution defined on the open interval J, where $0 \in J$. Suppose there exists $s \in J$ such that $\phi(s) \geq 1$. Without loss of generality, assume $s > 0$. Since $\phi(t)$ is continuous and $\phi(0) = \frac{1}{2}$, by the intermediate value theorem, there exists $s_0 \in (0,s)$ such that $\phi(s_0) = 1$. We can take s_0 to be the least value in $(0,s)$ such that $\phi(s_0) = 1$. That is to say, $\phi(t) < 1$ for all $0 < t < s_0$ and $\phi(s_0) = 1$.

We now consider the initial value problem

$$\frac{dx}{dt} = \frac{x^3 - x}{1 + t^2x^2} \quad \text{where } x(s_0) = 1.$$

The function $f(t,x) = \frac{x^3 - x}{1+t^2x^2}$ satisfies Picard's theorem on existence and uniqueness (Theorem 6.1), so there exists a unique solution defined on an interval $I = [s_0 - \alpha, s_0 + \alpha]$ for some $\alpha > 0$. The function $\phi(t)$ defined on J is a solution to this initial value probm, and it has the property that $\phi(t) < 1$ for all $t < s_0$. However, $\phi(t) = 1$ is a solution to this intiial value problem on I, but φ and ϕ are two different solutions to the initial value problem, contradicting the uniqueness of the solution. It follows that $\phi(t) < 1$ for all $t \in J$. Similarly, $\phi(t) > 0$ for all $t \in J$. □